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A General Class of Problems in Approximation.*

BY DUNHAM JACKSON.

1. *Introduction.* In a number of recent papers,† the author has discussed the existence and uniqueness of a polynomial or finite trigonometric sum affording the best approximation to a given function $f(x)$, in the sense of the method of least m th powers, and the convergence of the approximating sum to the value of $f(x)$ as its order is indefinitely increased. It has been apparent at almost every stage that the treatment was capable of very wide generalization, but the range of attention has been for the most part deliberately restricted, when it seemed that the presentation would gain in concreteness thereby. It is the purpose of this paper to bring out the considerations which give direction to one of the numerous possible forms of generalization. The central idea is the replacement of the m th power by a more general function of the error.

2. *Existence of an approximating function.* The study will be based on the following hypotheses, to which others will be added, in later sections of the paper, as occasion requires:

HYPOTHESIS A. *Let $E(y)$ be a function which is defined and continuous for all real values of y , and which becomes positively infinite as y goes to infinity in either direction.*

HYPOTHESIS B. *Let $p_1(x), \dots, p_n(x)$ be n functions of x , continuous and linearly independent for $a \leq x \leq b$, and let*

$$\phi(x) = c_1 p_1(x) + c_2 p_2(x) + \dots + c_n p_n(x)$$

*be an arbitrary linear combination of the p 's with constant coefficients.**

* Presented to the American Mathematical Society, December 29, 1922.

† The following papers will be cited by Roman numerals, as indicated: I. "On Functions of Closest Approximation," *Transactions of the American Mathematical Society*, Vol. 22 (1921), pp. 117-128; II. "On the Convergence of Certain Trigonometric and Polynomial Approximations," the same *Transactions*, Vol. 22 (1921), pp. 158-166; III. "Note on a Class of Polynomials of Approximation," the same *Transactions*, Vol. 22 (1921), pp. 320-326; IV. "Note on an Ambiguous Case of Approximation," the same *Transactions*, Vol. 25 (1923), pp. 333-337.

* For the special cases of polynomial or trigonometric approximation, the p 's would be powers of x or sines and cosines of multiples of x . The more general form of ϕ was extensively used in the earlier papers cited; the present generalization, as indicated above, relates to the function E .

HYPOTHESIS C. Let $f(x)$ be a continuous function of x in the interval $a \leq x \leq b$.

We shall inquire whether it is possible to determine the coefficients c_1, \dots, c_n , so as to reduce the value of the integral

$$J = J(\phi) = \int_a^b E[f(x) - \phi(x)] dx$$

to a minimum, the functions E, p_1, \dots, p_n , and f being given. It can be shown that there will always be at least one such determination of the coefficients, under the hypotheses stated.

Since $E(y)$ is continuous and becomes positively infinite as y becomes positively or negatively infinite, it has a minimum value l . Then it is certain that $J \geq l(b-a)$, and hence J has a finite lower limit for all possible choices of the coefficients c . The proof that this lower limit can be attained follows the lines of the corresponding proof in the paper IV; the differences, however, are perhaps sufficient to justify giving it in outline.

Consider first the integral

$$J_1 = \int_a^b E[\phi(x)] dx.$$

It will be shown that to every number H there corresponds a number Q , such that if the relation $J_1 \leq H$ is satisfied, it can be inferred that $|c_k| \leq Q$, $k = 1, 2, \dots, n$. The value of Q will depend on H , and on the functions E and p_1, \dots, p_n , and so, by implication, on the interval (a, b) in which the latter functions are assumed to be defined, but will be independent of the c 's.

By a lemma due to Sibirani,* there exists a constant P , independent of $c_1 \dots c_n$, such that if H_1 is the maximum of $|\phi(x)|$, it is certain that the greatest of the numbers $|c_k|$ can not exceed PH_1 . In other words, if c is the greatest value of $|c_k|$, then $H_1 \geq c/P$. Because of the uniform continuity of the p 's, there is a number $d > 0$ such that

$$|p_k(x') - p_k(x'')| \leq 1/(2nP), \quad k = 1, 2, \dots, n,$$

whenever $|x' - x''| \leq d$. This d is independent of the c 's, from the manner of its definition; and it may be assumed that $d < \frac{1}{2}(b-a)$. Since, in each of the n terms of the sum $\phi(x)$, $|c_k| \leq PH_1$, it is certain that

$$|\phi(x') - \phi(x'')| \leq H_1/2$$

for $|x' - x''| \leq d$. If x_0 is a point at which $\phi(x)$ takes on the value H_1 , it can be inferred that

$$(1) \quad |\phi(x)| \geq H_1/2$$

* Cf., e. g., loc. cit. I, Lemma I.

as long as $|x - x_0| \leq d$. The values of x satisfying the last requirement will surely belong to the interval (a, b) , on one side of x_0 at least, because of the assumption that $d < \frac{1}{2}(b - a)$, and so it is established that the relation (1) holds throughout an interval of length at least d .

When y is positive, let $E_1(y)$ be the minimum of $E(y')$ for $|y'| \geq y$. The hypothesis that $\lim_{y \rightarrow \infty} E(y) = +\infty$ implies that $\lim_{y \rightarrow \infty} E_1(y) = +\infty$; and $E_1(y)$ is a monotone function. It is seen that $E[\phi(x)] \geq E_1(H_1/2)$ throughout an interval of length at least d , while $E[\phi(x)] \geq l$ everywhere. If l_1 is the smaller algebraically of the numbers 0, l , the value of $\int E[\phi(x)] dx$, extended over any part of the interval (a, b) , will be $\geq l_1(b - a)$, and

$$J_1 \geq d.E_1(H_1/2) + l_1(b - a) \geq d.E_1[c/(2P)] + l_1(b - a).$$

In the last expression, everything but c is independent of the coefficients c_k ; it is recognized that J_1 becomes positively infinite as c becomes infinite. For every H , then, there is a Q such that $J_1 > H$ if $c > Q$, or, from the other point of view, $c \leq Q$ if $J_1 \leq H$.

Return now to the integral J . Let it be assumed for the moment that $f(x)$ is linearly independent of the functions p_1, \dots, p_n . Then $f - \phi$ may be regarded as a linear combination of the $n + 1$ linearly independent functions f, p_1, \dots, p_n , and the reasoning of the preceding paragraphs with regard to the integral J_1 is applicable to J . Any determination of the coefficients c_k which brings the value of J near its lower limit must belong to a restricted region in the space of n dimensions having the c 's for coördinates, this region may be regarded as including its boundary, and in the closed region so obtained there must be a determination of the c 's for which the integral J , being a continuous function of these parameters, reaches its lower limit.

If it happens that f is linearly dependent on p_1, \dots, p_n , then $f - \phi$ is a linear combination of the p 's. The coefficients by which $f - \phi$ is expressed in terms of the p 's are restricted in value if J is brought near its lower limit, the coefficients in $(-\phi)$ differ from these only by the values of the uniquely determined coefficients which serve to express f in terms of the p 's, and so the coefficients c_k which come into consideration belong once more to a restricted domain. It is seen that in this case also a suitable choice of the c 's will reduce J to a minimum.

A function ϕ for which the integral J takes on its minimum value will be called for brevity an *approximating function* for $f(x)$. The conclusion may be stated thus:

THEOREM I. *Under the conditions of Hypotheses A, B, and C, there exists at least one approximating function.*

3. *Properties of the approximating function.* Simple examples show * that the approximating function whose existence has been proved in the preceding section is not necessarily uniquely determined, unless further restrictions are placed on the function $E(y)$. It will be assumed throughout that $E(y)$ satisfies Hypothesis A of § 2. For the moment, the following additional requirement will be imposed:

HYPOTHESIS D'. Let $E(y)$ be such that

$$(2) \quad E\left(\frac{y_1 + y_2}{2}\right) < \frac{E(y_1) + E(y_2)}{2}$$

whenever y_1 and y_2 are distinct.

This is of course a material restriction. It may be said to characterize the function $E(y)$ as *properly convex*, in distinction from the usage which designates as *convex* † a function satisfying the same relation with the sign $<$ replaced by \leq , subject merely to the requirement that the equality shall not hold universally. It is a simple matter to prove

THEOREM II. If Hypotheses A, B, C, and D' are satisfied, the approximating function is unique.

Let γ be the minimum value of the integral J , and suppose, if possible, that ϕ_1 and ϕ_2 are two functions of the form ϕ , for each of which $J = \gamma$. Let $\phi_3 = \frac{1}{2}(\phi_1 + \phi_2)$. Then

$$f - \phi_3 = \frac{1}{2}[(f - \phi_1) + (f - \phi_2)],$$

and so, in the present situation,

$$E(f - \phi_3) < \frac{1}{2}[E(f - \phi_1) + E(f - \phi_2)]$$

at all points where $\phi_1 \neq \phi_2$. If ϕ_1 and ϕ_2 are not identical, it follows that $\int E(f - \phi_3) dx < \gamma$, which is contrary to the definition of γ .

For most purposes in this paper, Hypothesis D' can be replaced by the slightly less stringent

HYPOTHESIS D. Let $E(y)$ be such that

$$(3) \quad E\left(\frac{y_1 + y_2}{2}\right) \leq \frac{E(y_1) + E(y_2)}{2}$$

for all values of y_1 and y_2 . ‡

* Cf. the paper IV, § 3.

† See J. L. W. V. Jensen, "Sur les fonctions convexes, etc.", *Acta Mathematica*, Vol. 30 (1906), pp. 175-193.

‡ This hypothesis would not have been sufficient in the preceding theorem, either

From (3) it can be inferred more generally that if t is any number of the interval $0 < t < 1$, and if $y_t = ty_1 + (1-t)y_2$, then *

$$(4) \quad E(y_t) \leq tE(y_1) + (1-t)E(y_2).$$

Let ϕ_0 be the approximating function, or, if the determination is not unique, an approximating function, so that $J(\phi_0) = \gamma$, and let ϕ_1 be any function of the form ϕ for which $J(\phi_1) > \gamma$. Let $\phi_t = t\phi_1 + (1-t)\phi_0$. As a result of (4),

$$E(f - \phi_t) \leq tE(f - \phi_1) + (1-t)E(f - \phi_0),$$

and

$$(5) \quad J(\phi_t) \leq tJ(\phi_1) + (1-t)J(\phi_0) < J(\phi_1),$$

the last inequality being a consequence of the fact that $J(\phi_0) < J(\phi_1)$. Since t can be taken arbitrarily near to 1, the relation just obtained means that there exist functions ϕ_t with coefficients arbitrarily close to those of ϕ_1 , for which $J(\phi_t) < J(\phi_1)$, or, in other words: †

THEOREM III. *If Hypotheses A, B, C, and D are satisfied, no function ϕ can give a relative minimum for J , other than the absolute minimum $J = \gamma$.*

Now let the requirement (3)^o be temporarily waived altogether, to make way for the following hypothesis and conclusion:

HYPOTHESIS E. *Let $E(y)$ possess everywhere a continuous derivative with regard to y .*

THEOREM IV. *If Hypotheses A, B, C, and E are satisfied, and if ϕ_0 is an approximating function, it must necessarily be that*

$$(6) \quad \int_a^b E'(f - \phi_0) \phi(x) dx = 0$$

for every function ϕ which is a linear combination of the p 's.

for the proof or for the correctness of the conclusion; cf. the paper III, second footnote to § 1.

* If t is an integral multiple of $\frac{1}{2}N$, for some integral value of N , the fact is recognized almost immediately, as a result of successive bisections of the interval (y_1, y_2) . As $E(y)$ is continuous, a passage to the limit gives the result for an arbitrary value of t . Cf. Jensen, loc. cit., p. 180, and the paper of Hölder cited by Jensen, loc. cit., p. 192; also Galvani, "Sulle funzioni convesse di una o due variabili, definite in un aggregato qualunque," *Rendiconti del Circolo Matematico di Palermo*, Vol. 41 (1916), pp. 103-134, especially pp. 120-121.

† Cf. the paper I, p. 122.

Let ϕ be any such function, and let $\phi_t(x) = \phi_0(x) + t\phi(x)$. Then the integral $J(\phi_t)$ becomes a function of the parameter t , which is not restricted to the interval $(0, 1)$, as previously, but may take on arbitrary real values. All the requisite conditions of continuity being satisfied, $J(\phi_t)$ has a derivative with regard to t , which is given by the formula

$$d/dt J(\phi_t) = \int_a^b \partial/\partial t E(f - \phi_0 - t\phi) dx = - \int_a^b E'(f - \phi_0 - t\phi) \phi(x) dx.$$

If ϕ_0 is to have the minimizing property, $J(\phi_t)$ must be a minimum for $t = 0$, and dJ/dt must vanish for $t = 0$, which means that (6) must be satisfied.

Restoration of the requirement (3) leads to

THEOREM V. *If Hypotheses A, B, C, D, and E are satisfied, the necessary condition of the preceding theorem is also sufficient.**

The theorem means that if ϕ_1 is any function of the form ϕ for which $J(\phi_1) > \gamma$, the condition is not satisfied by ϕ_1 . Let ϕ_0 be an approximating function, and let $\psi(x) = \phi_0(x) - \phi_1(x)$. It will appear that $\psi(x)$, a linear combination of the p 's, gives the integral $\int E'(f - \phi_1)\psi(x) dx$ a value different from zero.

Let $\phi_t = \phi_1 + (1 - t)\psi = t\phi_1 + (1 - t)\phi_0$. Then

$$(7) \quad d/dt J(\phi_t) = \int_a^b E'[f - \phi_1 - (1 - t)\psi] \psi(x) dx.$$

But from the relation (5), in which the notation agrees with that now used,

$$J(\phi_1) - J(\phi_t) \geq (1 - t)[J(\phi_1) - J(\phi_0)],$$

for $0 < t < 1$, and in the limit

$$[d/dt J(\phi_t)]_{t=1} = \lim_{t \rightarrow 1} \frac{J(\phi_1) - J(\phi_t)}{1 - t} \geq J(\phi_1) - J(\phi_0) > 0,$$

which proves that the right-hand member of (7) can not be zero for $t = 1$.

4. *Special theorem on uniqueness of the approximating polynomial.*†

The conclusion with regard to the uniqueness of the approximating function can be made somewhat more general, in respect to the properties of the function $E(y)$, if the functions p_1, \dots, p_n are specialized in the following manner:

* Cf. the paper I, § 7, for both necessity and sufficiency.

† See note at the beginning of § 6.

HYPOTHESIS B'. Let $p_1(x), \dots, p_n(x)$ be the functions $1, x, \dots, x^{n-1}$, so that the function ϕ of Hypothesis B is an arbitrary polynomial of degree $n - 1$.

In this case the inequality (2) can be replaced by (3), together with:

HYPOTHESIS F. Let $E(y)$ take on its minimum value l only for a single value of y , say for $y = a$.

It will be seen that something in the nature of the last requirement is indispensable, even with the present restriction on $\phi(x)$. For if $E(y) = l$ for two distinct values of y , it must be that $E(y) = l$ for all intermediate values of y , because of (3) and the fact that $E(y)$ can never be less than l . That is, $E(y)$ is equal to l throughout an interval. If $f(x) = 0$ identically, the approximating property will be possessed by any constant belonging to this interval, or by any polynomial with a constant term interior to the interval and with the remaining coefficients sufficiently small. On the other hand, the combination of Hypotheses A, D, and F with regard to $E(y)$ is not sufficient unless some limitation is placed on the original generality of the functions ϕ . This is shown by the example cited in the preceding section from the paper III, where $E(y)$ is equal to $|y|$.

There is no loss of generality, for the present purpose, in supposing that $l = a = 0$. For the addition of a constant to $E(y)$ has no influence on the determination of the approximating function, since it affects all values of the integral J alike; and an approximating function for $f(x)$ according to $E(y)$ is an approximating function for $f(x) - a$ according to $E(y + a)$. This is of course equally true, whether the functions ϕ under consideration are polynomials or not.

The working hypothesis is, then, that $E(y)$ satisfies A and D, vanishes for $y = 0$, and is positive for all other values of y , while $\phi(x)$ is an arbitrary polynomial of degree $n - 1$; $f(x)$, as always hitherto, is subject only to Hypothesis C.

If y_1 and y_2 are any two values of y which have opposite algebraic signs, the relation (3) is one of actual inequality; (2) is satisfied for such values of y_1 and y_2 . This is evident graphically, and the essentials of the geometric figure are readily translated into algebraic language. Suppose, if possible, that ϕ_0 and ϕ_1 are two distinct approximating polynomials, and let $\phi_2 = \frac{1}{2}(\phi_0 + \phi_1)$. It follows from what has just been said that there can be no

† If $E(y + a) = \bar{E}(y)$ and $f(x) - a = \bar{f}(x)$, then $\bar{E}(\bar{f} - \phi) = E(f - \phi)$.

value of x for which $f - \phi_0$ and $f - \phi_1$ have opposite signs. For such a value of x , and neighboring values of x , would make $E(f - \phi_2) < \frac{1}{2}[E(f - \phi_0) + E(f - \phi_1)]$, the relation being at most an equality elsewhere, and $J(\phi_2)$ would be less than $J(\phi_0)$ or $J(\phi_1)$, contrary to hypothesis. If $f - \phi_0 = r_0$ and $f - \phi_1 = r_1$, it must be that $r_0 r_1 \geq 0$ throughout (a, b) . This means further that in every interval where either remainder changes sign, there must be at least one point where both remainders vanish simultaneously, a point, consequently, where ϕ_0 and ϕ_1 , being both equal to f , are equal to each other.* The function $\phi_t = \phi_0 + t(\phi_1 - \phi_0)$ is an approximating polynomial for any value of t in the interval $(0, 1)$, because of (3) and its corollary (4), and the reasoning that has been applied to ϕ_0 and ϕ_1 is valid for the pair of functions ϕ_t corresponding to any two values of t in this interval.

The proof of uniqueness is readily completed, if it is supposed, in addition to the preceding assumptions, that $E(y)$ has a continuous derivative (Hypothesis E).† The approximating polynomial ϕ_0 is then subject to the restriction (6) of the preceding section, for every polynomial $\phi(x)$ of degree $n - 1$. This means that $E'(f - \phi_0)$, if not identically zero, must change sign at least n times as x goes from a to b . For if there were fewer changes of sign, it would be possible to construct a polynomial of degree $n - 1$ having the same sign as $E'(f - \phi_0)$ at all points where the latter function is different from zero,‡ and (6) would be violated. In consequence of the properties that have been assigned to $E(y)$, it is evident geometrically, and easily proved analytically § by the use of (4), that $E'(y)$ always has the same sign as y and vanishes only for $y = 0$. If $E'(f - \phi_0)$ vanishes identically, then, ϕ_0 is identically equal to f , and is clearly unique as an approximating polynomial. If $E'(f - \phi_0)$ is not identically zero, the remainder $f - \phi_0$ itself

* Cf. the paper III, § 3. This conclusion holds, without change in the method of proof, for the more general form of ϕ .

† Even this would not be enough to make the approximating function unique, under the more general hypothesis as to the form of ϕ . Let $E(y) = |y|$ for $|y| \geq \delta$, $0 < \delta < 1$, the definition of $E(y)$ for $|y| \leq \delta$ being supplied in any one of an infinite variety of possible ways so that the requirements of convexity and differentiability are satisfied. This $E(y)$ will not vanish for $y = 0$, but it has been mentioned that that is of no consequence for the point at issue. With the method of construction indicated, it can evidently be assumed, and in fact must necessarily be true, that $E(y) \geq |y|$ throughout. Let the interval (a, b) be $(-1, 1)$, let $f(x) = 1$, and let $n = 1$, $p_1(x) = x$, so that $\phi(x) = c_1 x$ is still a polynomial, but not a polynomial of degree $n - 1$. The value of the integral J can not possibly be less than the minimum that would be obtained if $E(y)$ were replaced by $|y|$, and this value of J is actually reached, with the present $E(y)$, if c_1 has any value between $-(1 - \delta)$ and $1 - \delta$.

‡ Cf. the paper I, p. 126.

§ Cf. the discussion of $dJ(\phi_t)/dt$ at the end of the preceding section.

must change sign n times in (a, b) . By the reasoning of the preceding paragraph, if ϕ_1 is alleged to be another approximating polynomial, ϕ_1 must be equal to ϕ_0 for n points in (a, b) , and so can not be distinct from ϕ_0 .

The hypothesis as to the existence of $E'(y)$ was used in the above reasoning only to establish the fact that $f - \phi_0$ must change sign n times (if not identically zero). If $f - \phi_0$ does change sign n times, any approximating polynomial must be equal to ϕ_0 at n points, and so must be identical with ϕ_0 , whether $E'(y)$ exists or not. That is, if there exists an approximating polynomial for which the remainder changes sign n times, there can be no other approximating polynomial. If ϕ_0 and ϕ_1 are supposed to be two approximating polynomials, $\phi_t = \phi_0 + t(\phi_1 - \phi_0)$ will be an approximating polynomial for any value of t in $(0, 1)$, as has already been pointed out. So it remains only to investigate the possibility of the existence of two approximating polynomials ϕ_0 and ϕ_1 , such that $r_t = f - \phi_t$ does not change sign more than $n - 1$ times in (a, b) for any value of t belonging to the interval $0 \leq t \leq 1$. Furthermore, it may be assumed that neither r_0 nor r_1 is identically zero, because if f were identically equal to a polynomial of degree $n - 1$, this would certainly be the only approximating polynomial.

Let L be a positive number, which may be chosen once for all; it would be sufficient to take $L = 1$. Let M be the larger of the numbers $E(L)/L$, $E(-L)/L$. Then, because of (4), $E(y) \leq M|y|$ for $|y| \leq L$. There will be occasion in the course of the proof to make use of this fact.

It will be convenient to complete the proof first for the extreme case that $r_0 \geq 0$ and $r_1 \geq 0$ throughout (a, b) , so that r_t , being equal to $r_0 + t(r_1 - r_0)$, and so intermediate between r_0 and r_1 , does not change sign at all for any value of t in $(0, 1)$. Apart from the values of x for which r_0 and r_1 may vanish simultaneously, it is conceivable that there may be points at which r_0 and r_1 are equal to each other without being equal to zero. There can not be more than a finite number of such points, for a point of equality of the remainders r_0 and r_1 is a point of equality of the polynomials ϕ_0 and ϕ_1 . Hence there can not be more than a finite number of positive values taken on by r_0 and r_1 simultaneously. Let L_1 be the smallest of these values, if any exist. Otherwise, let L_1 be the largest value taken on by either r_0 or r_1 separately. It will be shown that $E(y)$ must be linear for $0 \leq y \leq L_1$, if the assumption that ϕ_0 and ϕ_1 are distinct approximating polynomials is to be even temporarily maintained.

If, in the relation

$$E(r_t) \leq tE(r_1) + (1-t)E(r_0)$$

(cf. (4)), the sign of inequality were to hold for a value of t in $(0, 1)$ and a value of x in (a, b) , the integral $J(\phi_t)$, formed for the value of t in question, would be smaller than $J(\phi_0)$ or $J(\phi_1)$, and the hypothesis with regard to ϕ_0 and ϕ_1 would be contradicted. It must be that the relation is an equality throughout, or, in other words, that $E(y)$ is linear for $y_1 \leq y \leq y_2$, if y_1 and y_2 are the smaller and the larger respectively of the values taken on by r_0 and r_1 at any one point of (a, b) .

Let η be any number belonging to the interval $0 < \eta < L_1$. For each value of x in (a, b) , let $\rho_0(x)$ be the smaller of the two numbers $r_0(x)$, $r_1(x)$, and let $\rho_1(x)$ be the larger. From the definition of L_1 , there will be one or more values of x for which $\rho_1 = \eta$, and among these there will be at least one, $x = x'$, in the immediate neighborhood of which ρ_1 takes on values larger* than η . Since $\eta < L_1$, it is certain that $\rho_0(x') < \eta$; for $\rho_0 \leq \rho_1$, and ρ_0 and ρ_1 do not take on simultaneously any value smaller than L_1 . Throughout a sufficiently restricted neighborhood of x' , ρ_0 will remain less than η , and as there are points of this neighborhood at which $\rho_1 > \eta$, there will surely exist a point x'' , such that $\rho_0(x'') < \eta < \rho_1(x'')$. By the preceding paragraph, this means that η , regarded as a value of y , is interior to an interval throughout which $E(y)$ is linear. If $\epsilon > 0$, every value of y from ϵ to $L_1 - \epsilon$, extremes included, is interior to such an interval, and so, by the Heine-Borel theorem, interior to one of a finite number of such intervals. But if $E(y)$ is linear throughout each of a finite number of intervals, overlapping as these must overlap, it must have a single linear expression over the whole range covered by them, and so, in the present instance, over the whole range from ϵ to $L_1 - \epsilon$. Since ϵ is arbitrarily small, it follows by continuity that $E(y)$ is linear for $0 \leq y \leq L_1$, as previously asserted. If μ is the value of the constant derivative, $E(y) = \mu y$ in $(0, L_1)$.

This is important for the corollary that if $0 \leq y_1 \leq y_2$, then

$$(8) \quad E(y_2) - E(y_1) \geq \mu(y_2 - y_1).$$

For if the interval from 0 to y_2 is divided into 2^N equal parts, the increment of $E(y)$ in any one of these parts is less than or equal to the increment in the next succeeding one, as a direct consequence of (3), and so less than or equal to the increment in any succeeding one; the increment in the first of the 2^N parts is $\mu y_2 / 2^N$, if N is large enough to make $y_2 / 2^N < L_1$; (8) is thus

* Let η_1, η_2, \dots , be an infinite sequence of numbers such that $\eta < \eta_k < L_1$ for all values of k , and $\lim_{k \rightarrow \infty} \eta_k = \eta$. For each k there will be an x_k such that $\rho_1(x_k) = \eta_k$. The points x_k must have at least one limit point, and by the continuity of ρ_1 this will be a point at which $\rho_1 = \eta$.

seen to be true if y_1 is an integral multiple of $y_2/2^N$; and the general conclusion is obtained by a passage to the limit.*

Now let ϵ be any positive quantity. For any particular value of t in $(0, 1)$, let R_1 be the set of points in (a, b) at which $r_t \geq \epsilon$. The discussion is still restricted to the special case that $r_0 \geq 0$ and $r_1 \geq 0$ throughout, so that the same is true of r_t . For any value of x in R_1 , the numbers $r_t(x) - \epsilon$ and $r_t(x)$ are in the relation of y_1 and y_2 in the preceding paragraph, and

$$E(r_t) - E(r_t - \epsilon) \geq \mu\epsilon, \quad E(r_t - \epsilon) \leq E(r_t) - \mu\epsilon.$$

In the complementary set S_1 , where $r_t < \epsilon$, $|r_t - \epsilon| \leq \epsilon$. As ϵ is presently to be allowed to approach zero, it may be assumed that $\epsilon \leq L$, and then

$$E(r_t - \epsilon) \leq M\epsilon$$

throughout S_1 . Hence

$$\begin{aligned} \int_a^b E(r_t - \epsilon) dx &= \int_{R_1} + \int_{S_1} \leq \int_{R_1} E(r_t) dx - \mu\epsilon mR_1 + M\epsilon mS_1 \\ &\leq \int_a^b E(r_t) dx - \epsilon(\mu mR_1 - MmS_1), \end{aligned}$$

the symbol m standing for *measure of*. The first member of this continued relation is the integral J , formed for the polynomial $\phi_t + \epsilon$, while the first term of the last member is $J(\phi_t)$. The relation $J(\phi_t + \epsilon) < J(\phi_t)$ would be inconsistent with the character of ϕ_t as approximating polynomial. Therefore, in order that this inequality shall not hold, it must be that $MmS_1 \geq \mu mR_1$. As ϵ may be taken arbitrarily small, it follows that in the limit \dagger $MmS \geq \mu mR$, if S is the set of points where $r_t = 0$ and R the set of points where $r_t > 0$. Since $mR = b - a - mS$, this means that

$$mS \geq \mu(b - a)/(M + \mu).$$

The reasoning is valid for any value of t in $(0, 1)$; and the positive quantity $h = \mu(b - a)/(M + \mu)$ is independent of t . So it has been inferred that $r_t = 0$, or $\phi_t = f$, throughout a set of measure at least h , for every value of t in $(0, 1)$. But this is manifestly impossible, for the sets S corresponding to any two different values of t are mutually exclusive, except for a finite number of points at most, a point common to two different sets S being a point at which two different polynomials ϕ_t of degree $n - 1$ have a common value. It must be concluded that the assumption that ϕ_0 and ϕ_1 are

* Cf. Galvani, loc. cit., p. 106.

† Cf. the paper III, § 4.

two distinct approximating polynomials, such that $r_0 \geq 0$ and $r_1 \geq 0$ throughout (a, b) , is inadmissible.

If it is assumed that $r_0 \leq 0$ and $r_1 \leq 0$ throughout, a contradiction is obtained in a similar fashion.

Finally, attention must be directed to the hypothesis that the range of variation of r_0 and r_1 includes both positive and negative values.* In this case it appears that there is a positive number L_1 such that $E(y)$ is linear for $0 \leq y \leq L_1$, and a positive number L_2 such that $E(y)$ is linear for $-L_2 \leq y \leq 0$. If the constant slopes in these intervals are μ_1 and $-\mu_2$ respectively, the relations

$$E(y_2) - E(y_1) \geq \mu_1(y_2 - y_1), \quad E(y_1) - E(y_2) \geq \mu_2(y_2 - y_1),$$

hold respectively for $0 \leq y_1 \leq y_2$ and for $y_1 \leq y_2 \leq 0$. Let μ be the smaller of the positive numbers μ_1, μ_2 . Then

$$(9) \quad |E(y_2) - E(y_1)| \geq \mu |y_2 - y_1|$$

whenever $y_1 y_2 \geq 0$.

Let ϵ once more be an arbitrary positive quantity. For any particular value of t in $(0, 1)$, let q be the number of changes of sign of $r_t(x)$ in (a, b) . The present hypothesis is that $q \leq n - 1$ for every t , the possibility that $q \geq n$ for even a single value of t having been already disposed of. It is possible to construct a polynomial $\dagger \psi(x)$, of degree q , such that $r_t \psi \geq 0$ throughout (a, b) . The constant factor in ψ which is left arbitrary by the last-named requirement will be supposed determined so that the maximum of $|\psi|$ in (a, b) is 1. If r_t does not change sign at all, for the particular value of t in question, it will be understood that $\psi = \pm 1$ identically, the sign of ψ being that of r_t . The further course of the proof involves an investigation of the value of the integral J corresponding to the polynomial $\phi_t + \epsilon\psi$, the corresponding remainder being $r_t - \epsilon\psi$.

Let R_1 be the set of points in (a, b) at which $|r_t| \geq \epsilon$, and S_1 the complementary set, where $|r_t| < \epsilon$. In R_1 , $r_t(r_t - \epsilon\psi) \geq 0$, whence, by (9),

$$E(r_t) - E(r_t - \epsilon\psi) \geq \mu\epsilon |\psi|, \quad E(r_t - \epsilon\psi) \leq E(r_t) - \mu\epsilon |\psi|.$$

At a point of S_1 , it may be that $|r_t| \geq \epsilon |\psi|$, or it may be that $|r_t| \leq \epsilon |\psi|$. In the former case, $E(r_t - \epsilon\psi) \leq E(r_t)$. In the latter case,

* That is, at least one of the two remainders changes sign in (a, b) , or else one of them takes on positive values for some values of x and the other becomes negative for other values of x . It has already been seen that they can not have opposite signs for the same x .

† For this whole demonstration, cf. the paper III, § 4.

$|r_t - \epsilon\psi| \leq \epsilon|\psi|$, and if it is assumed that $\epsilon \leq L$, as may be done, inasmuch as ϵ is ultimately to approach zero, $E(r_t - \epsilon\psi) \leq M\epsilon|\psi|$. In either case, and so throughout S_1 ,

$$E(r_t - \epsilon\psi) \leq E(r_t) + M\epsilon|\psi|.$$

Consequently

$$\int_a^b E(r_t - \epsilon\psi) dx = \int_{R_1} + \int_{S_1} \leq \int_a^b E(r_t) dx - \mu \epsilon \int_{R_1} |\psi| dx + M\epsilon \int_{S_1} |\psi| dx.$$

The integral on the left is $J(\phi_t + \epsilon\psi)$, and the first integral on the right is $J(\phi_t)$. As the relation $J(\phi_t + \epsilon\psi) < J(\phi_t)$ is inadmissible, it must be that

$$M \int_{S_1} |\psi| dx \geq \mu \int_{R_1} |\psi| dx.$$

This is true for all values of ϵ , provided merely that $\epsilon \leq L$. If S is the set of points where $r_t = 0$ and R the set where $|r_t| > 0$, it follows that in the limit, as ϵ approaches 0,

$$M \int_S |\psi| dx \geq \mu \int_R |\psi| dx.$$

By addition of the same quantity to both sides, the last relation can be transformed into

$$(10) \quad (M + \mu) \int_S |\psi| dx \geq \mu \int_a^b |\psi| dx.$$

Let the coefficient of x^{k-1} in ψ be denoted by c'_k , so that $\psi(x) = c'_1 + c'_2x + \cdots + c'_nx^{n-1}$, and let

$$\int_a^b |\psi| dx = G, \quad \omega(x) = c'_1(x-a) + c'_2(x^2-a^2)/2 + \cdots + c'_n(x^n-a^n)/n.$$

Then

$$|\omega(x)| = \left| \int_a^x \psi(x) dx \right| \leq \int_a^x |\psi(x)| dx \leq G.$$

But $\omega(x)$ may be regarded as a linear combination of the n linearly independent functions $(x^k - a^k)/k$, $k = 1, 2, \dots, n$. By the lemma of Sibirani cited in § 2, there exists a constant, which may be denoted in this connection by Q_1 , independent of c'_1, \dots, c'_n , such that $|c'_k| \leq Q_1G$, $k = 1, 2, \dots, n$. If W is the greatest value of any of the quantities $1, |x|, \dots, |x^{n-1}|$ in (a, b) ,

$$|\psi(x)| \leq nQ_1GW, \quad \int_S |\psi| dx \leq nQ_1GWmS.$$

So (10) means that

$$\mu G \leq (M + \mu) \int_S |\psi| dx \leq (M + \mu)nQ_1GWmS,$$

$$mS \geq \frac{\mu}{(M + \mu)nQ_1W}.$$

The last relation must be true for every value of t in $(0, 1)$, and the quantity on the right, which may be denoted by h , is independent of t . It appears, then, on the basis of the hypothesis to be disproved, that r_t must be equal to 0, and ϕ_t equal to f , throughout a set of measure at least h , for every value of t in $(0, 1)$, which is impossible.

Thus is completed the proof which has occupied practically the whole of this section. To state the conclusion once more, it is this:

THEOREM VI. *If Hypotheses A, B', C, D, and F are satisfied, the approximating function is unique.*

The demonstration can be carried through, with minor changes, if $\phi(x)$ is a finite trigonometric sum, instead of a polynomial. More generally, instead of specifying the precise form of $\phi(x)$ at all, it would be possible to express its significant properties in terms of postulates supplementing those of the earlier sections, as has been done in the case of $E(y)$. But this line of inquiry will not be pursued further here.

5. *Continuity of variation of the approximating function.** For the next stage in the discussion, let it be assumed that $E(y)$ and the functions $p_k(x)$ satisfy Hypotheses A and B respectively, and furthermore that they have such properties that the approximating function for an arbitrary continuous $f(x)$ is uniquely determined, whether as a result of the hypotheses considered in § 3 and § 4, or independently of those particular hypotheses. It is readily seen that the dependence of the approximating function on $f(x)$ is not linear, in general, but it will be shown that the dependence is continuous, in the sense of the following assertion.

Let f_0 be a given continuous function in (a, b) , and ϕ_0 its approximating function. For every positive ϵ there is a positive δ , such that if F is any continuous function differing from f_0 by less than δ throughout (a, b) , and Φ its approximating function, then no coefficient in Φ can differ from the corresponding coefficient in ϕ_0 by more than ϵ .

Since the only thing required of δ is that it be sufficiently small, it can be assumed throughout that $\delta \leq 1$. Then, if U is the maximum of $|f_0|$, it is certain that $|F| < U + 1$ throughout (a, b) , for all functions F that come into consideration. There will be occasion to use the lemma that the corresponding functions Φ also are uniformly bounded. It was shown in § 2 that if ϕ is any linear combination of the functions p_k , and if H_1 is the maxi-

* This topic was not discussed in the earlier papers that have been cited.

sum of $|\phi|$, then $|\phi| \geq H_1/2$ throughout an interval of length at least d , where d is a positive number independent of the coefficients in ϕ . Suppose $H_1 \geq 4(U+1)$. Then $|\phi - F| \geq (H_1/2) - (U+1) \geq H_1/4$, throughout an interval of length at least d , for every function F of the class under discussion. It follows that for every number Y there is a number Z , such that

$$\int_a^b E(F - \phi) dx > Y$$

if $H_1 > Z$. On the other hand, when Φ is the approximating function for F , it must be that

$$\int_a^b E(F - \Phi) dx \leq \int_a^b E(F) dx,$$

for Φ must give a smaller value to the integral than any other function ϕ which is a linear combination of the p_k 's, and $\phi = 0$ is such a function. But the value of the integral on the right is bounded, for any class of uniformly bounded functions F . If Y_1 is an upper bound for the integral when $|F| < U+1$, and if Z_1 is the value of Z corresponding to $Y = Y_1$, it is certain that $H_1 \leq Z_1$ whenever ϕ is one of the approximating functions Φ . This has the further implication, by the fundamental lemma of Sibirani, that the coefficients in the functions Φ are uniformly bounded.

For the purposes of the present demonstration, the more explicit notation $J(f, \phi)$ will be used in place of $J(\phi)$ or simply J , because there is occasion now to consider different functions f simultaneously. The argument of the function $E(y)$ can be restricted throughout to the range $|y| \leq U+1+Z_1$, where U and Z_1 are defined as in the preceding paragraph. In the closed interval specified, $E(y)$ is uniformly continuous. Let η be an arbitrary positive quantity. There will be a positive number ξ , which may be supposed ≤ 1 , such that $|E(y') - E(y'')| \leq \eta/[2(b-a)]$ if $|y'| \leq U+1+Z_1$, $|y''| \leq U+1+Z_1$, and $|y' - y''| \leq \xi$. Let F be a continuous function such that $|F - f_0| \leq \xi$ throughout (a, b) . Then

$$|J(F, \phi_0) - J(f_0, \phi_0)| = \left| \int_a^b [E(F - \phi_0) - E(f_0 - \phi_0)] dx \right| \leq \eta/2.$$

Similarly,

$$|J(F, \Phi) - J(f_0, \Phi)| \leq \eta/2.$$

Since ϕ_0 and Φ are the approximating functions for f_0 and F respectively,

$$J(f_0, \Phi) \geq J(f_0, \phi_0), \quad J(F, \phi_0) \geq J(F, \Phi).$$

By suitable combination of these inequalities it is found that

$$J(f_0, \phi_0) \leq J(f_0, \Phi) \leq J(F, \Phi) + \eta/2 \leq J(F, \phi_0) + \eta/2 \leq J(f_0, \phi_0) + \eta.$$

the significant fact being that

$$0 \leq J(f_0, \Phi) - J(f_0, \phi_0) \leq \eta.$$

That is, the difference between $J(f_0, \Phi)$ and $J(f_0, \phi_0)$ can be brought arbitrarily near to zero by taking ξ sufficiently small.

Now suppose the main theorem of the section were not true; it will be shown that this leads to a contradiction. Let $\delta_1, \delta_2, \dots, \delta_i, \dots$ be a diminishing sequence of positive quantities approaching zero. By the assumption which is to be disproved, there exists for each i a continuous function F_i , such that $|F_i - f_0| < \delta_i$ throughout (a, b) , and such that some coefficient in the approximating function Φ_i that belongs to F_i differs from the corresponding coefficient in ϕ_0 by more than ϵ , where ϵ is some positive quantity independent of i . Let the coefficients of Φ_i be regarded as the coördinates of a point in n -dimensional space. It has been shown that these points will lie in a bounded region, and they must have at least one limit point.* Let ϕ_1 stand for a linear combination of the p 's with the coördinates of such a limit point as coefficients. From the uniform continuity of $E(y)$, for the values of its argument that come into consideration, it can be inferred that

$$\lim'_{i=\infty} J(f_0, \Phi_i) = J(f_0, \phi_1),$$

where the accent indicates that i is to range only over the indices belonging to a partial sequence having no other limit point than the one in question. But the result of the preceding paragraph means that

$$\lim'_{i=\infty} J(f_0, \Phi_i) = J(f_0, \phi_0)$$

(the latter relation being equally true if the accent is omitted). That is, $J(f_0, \phi_1) = J(f_0, \phi_0)$, and ϕ_1 is an approximating function for f_0 . The function ϕ_1 can not be identical with ϕ_0 , because at least one of its coefficients differs from the corresponding coefficient in ϕ_0 by ϵ or more. So the desired contradiction has been obtained, under the original hypothesis that the approximating function is unique, and the theorem is proved. It can be stated thus:

* It is not necessary to inquire whether there will certainly be infinitely many *distinct* sets of coefficients; it will serve the purpose to understand that if infinitely many set^s of coefficients are identical, the corresponding point is to be regarded in this connection as a limit point of the sequence.

THEOREM VII. *If Hypotheses A, B, and C are satisfied, and if the approximating function for an arbitrary continuous $f(x)$ is unique, its dependence on $f(x)$ is continuous, in the sense explained at the beginning of the section.*

6. *Convergence of the approximating function as n becomes infinite.**

In this section, [as in § 4,] it will be necessary to restrict the generality of the functions p_k , and [here again] no attempt will be made to minimize the restrictions in this particular. It will be assumed quite specifically [not, as before, that $\phi(x)$ is a polynomial, but] that $\phi(x)$ is a finite trigonometric sum. More definitely, the previous notation will be changed to the extent of writing n' for the number of functions in the set of p 's, instead of n , and it will be assumed that n' is an odd number, $n' = 2n + 1$; then the specification becomes

HYPOTHESIS B''. *Let $p_1(x), \dots, p_{n'}(x)$, $n' = 2n + 1$, be the functions $1, \cos x, \dots, \cos nx, \sin x, \dots, \sin nx$, so that the linear combination corresponding to the function ϕ of Hypothesis B is an arbitrary trigonometric sum of the n th order.*

In this connection, an arbitrary $\phi(x)$ will be denoted by $t_n(x)$, and an approximating function for a given function $f(x)$ by $T_n(x)$. It will be supposed that the interval (a, b) is of length 2π . The problem is the investigation of the convergence of $T_n(x)$ to the value of $f(x)$, as n becomes infinite.

With regard to the function $E(y)$, it will be assumed that it reaches its minimum only for a single value a of y , as the problem otherwise loses most of its interest. On the other hand, it is not necessary to assume that the approximating function $T_n(x)$ is uniquely determined,[†] and no such assumption will be made, either explicitly [as in § 5], or implicitly through the hypotheses of § 3 [or § 4]. The addition of a constant to $E(y)$ does not affect the determination of the approximating functions [as was pointed out in an early paragraph of § 4], and it may be supposed without loss of generality that the arbitrary constant is so determined that the minimum value of $E(y)$ is zero. The value of a has the following significance. Suppose it can be shown, when $a = 0$, and when suitable additional hypotheses are fulfilled, that $T_n(x)$ converges to the value of $f(x)$. If $E(y)$ is replaced by $E(y - a)$, the same $T_n(x)$, for a given n , will be an approximating function for $f(x) + a$,

* Except for the inessential words in brackets, this section can be read without reference to § 4 or § 5.

† Cf. the paper IV, § 4.

instead of $f(x)$, and as n becomes infinite $T_n(x)$, in approaching $f(x)$, will converge, not to the value of the function for which it gives the best "approximation," in the peculiar sense of the present discussion,* but to that function diminished by the constant α . With this indication of what is to be expected in the more general cases, it will be assumed henceforth that $\alpha = 0$.

The working hypothesis with regard to $E(y)$ is then that it satisfies Hypothesis A, vanishes for $y = 0$, and is positive for all other values of y .

Let $f(x)$ be a continuous function of period 2π . If $T_n(x)$ is an approximating function † for $f(x)$, and if $t_n(x)$ is an arbitrary trigonometric sum of the n th order, then $T_n(x) - t_n(x)$ will be an approximating function for $f(x) - t_n(x)$. For the aggregate of possible values of the integral J is the same in the case of $f - t_n$ as in the case of f ; the minimum value of J is therefore the same in both cases; and the integral of $E[(f - t_n) - (T_n - t_n)]$ is of course the same as the integral of $E(f - T_n)$. If $f - t_n$ is denoted by ψ_n , the difference between f and an approximating function for f will be the same as the difference between ψ_n and the approximating function, or a suitably chosen approximating function, for ψ_n . The convergence proof that is to be given below depends on this observation.

Let ϵ_n be the maximum of $|\psi_n|$. Let $\tau_n(x)$ be an approximating sum of the n th order for ψ_n , and let λ_n be the maximum of $|\tau_n|$. It will be convenient to suppose at first that $\lambda_n \geq 4\epsilon_n$, the contrary hypothesis being left for later consideration. By a theorem of S. Bernstein,‡ the derivative of τ_n can never exceed $n\lambda_n$ in absolute value. This means that a point at which $|\tau_n|$ takes on its maximum value λ_n will be contained in an interval of length at least $\frac{1}{n}$ (at least $1/(2n)$ on each side of the point in question) throughout which $|\tau_n| \geq \lambda_n/2$. Since $|\psi_n| \leq \epsilon_n \leq \lambda_n/4$, there will be an interval of length at least $1/n$ throughout which $|\psi_n - \tau_n| \geq \lambda_n/4$. If $E_1(y)$, as in § 2, denotes the minimum of $E(y')$ for $|y'| \geq y$, when y is positive, it is certain that

$$\gamma_n = \int_0^{2\pi} E(\psi_n - \tau_n) dx \geq 1/n E_1(\lambda_n/4).$$

* It requires perhaps a somewhat strained interpretation to use the words "approximating function" in this connection at all, but it has seemed better to retain the terms appropriate to the most important cases than to bring in new ones.

† For the entire convergence proof, cf. the paper II, §§ 2-4.

‡ Cf., e. g., C. de la Vallée Poussin, *Leçons sur l'approximation des fonctions d'une variable réelle*, Paris, 1919, pp. 39-42.

§ The problem under discussion being one of convergence for $n = \infty$, the value $n = 0$ may be left out of account.

On the other hand, since τ_n is an approximating sum for ψ_n , it is certain that the value of this integral would not be diminished if τ_n were replaced by any other trigonometric sum of order n , and, in particular by zero. If $E_2(y)$, defined for $y \geq 0$, is the maximum of $E(y)$ for $|y'| \leq y$, it must be that

$$\gamma_n \leq \int_0^{2\pi} E(\psi_n) dx \leq 2\pi E_2(\epsilon_n).$$

From the two inequalities that have been found for γ_n , it follows that

$$(11) \quad E_1(\lambda_n/4) \leq 2\pi n E_2(\epsilon_n).$$

This is on the hypothesis that $\lambda_n/4 \geq \epsilon_n$. If $\lambda_n/4 < \epsilon_n$, it appears in the first place that $E_1(\lambda_n/4) \leq E_1(\epsilon_n)$, since $E_1(y)$ is a non-decreasing function of y . Furthermore, $E_1(y) \leq E_2(y)$ for all values of y , because the value $y' = y$ is admitted in the definition of both functions. So $E_1(\epsilon_n) \leq E_2(\epsilon_n)$; and as $1 < 2\pi n$ for $n \geq 1$, the inequality (11) retains its validity, whatever the relative magnitudes of λ_n and ϵ_n may be.

The value of ϵ_n depends on the arbitrary trigonometric sum $t_n(x)$ which enters into the definition of ψ_n . It is itself not completely arbitrary, for a given value of n , because it can not be reduced to zero nor brought arbitrarily near to zero, unless $f(x)$ is identically a trigonometric sum of the n th order. The function f being of period 2π and continuous, it is known by Weierstrass's theorem that ϵ_n can be made to approach zero as n becomes infinite. The rapidity with which it can be made to approach zero depends on the properties of $f(x)$, in a manner which has been the subject of more or less extensive investigation.* Let it be assumed here that $f(x)$ is a function for which ϵ_n can be made to diminish so rapidly that $\lim_{n \rightarrow \infty} n E_2(\epsilon_n) = 0$. This hypothesis is necessarily somewhat indirect in its statement, as long as the function $E(y)$ retains so wide a range of generality. Its interpretation in terms of the properties of continuity, existence of derivatives, etc., of $f(x)$, when a particular $E(y)$ is specified, depends on the availability of appropriate theorems on trigonometric approximation; but perfectly definite sufficient conditions can be written down immediately in a large variety of cases.†

* Cf. e. g., D. Jackson, "The General Theory of Approximation by Polynomials and Trigonometric Sums," *Bulletin of the American Mathematical Society*, Vol. 27 (1920-21), pp. 415-431; see especially § 6.

† For example, if $E(y) \leq |y|$ throughout an interval including the point $y = 0$ in its interior, it is sufficient that $f(x)$ have a continuous derivative; if $E(y) \leq y^2$ throughout such an interval, it is sufficient that $\lim_{\delta \rightarrow 0} \omega(\delta)/\sqrt{\delta} = 0$, where $\omega(\delta)$ is the modulus of continuity of $f(x)$; etc. Cf. the paper II, § 4.

With the hypothesis that $\lim_{n \rightarrow \infty} nE_2(\epsilon_n) = 0$, let ϵ be an arbitrary positive quantity. It follows from (11) that $E_1(\lambda_n/4)$ approaches zero as n becomes infinite. But as $E_1(y)$ is continuous, becomes infinite with y , and vanishes only for $y = 0$, this means that λ_n itself approaches zero. So, as λ_n is the maximum of $|\tau_n|$, it is certain that $|\tau_n| < \epsilon/2$, for all values of x , if n is sufficiently large. On the other hand, n can be chosen so large as to make $\epsilon_n < \epsilon/2$, where ϵ_n , it will be remembered, is the maximum of $|\psi_n|$. Hence $|\psi_n - \tau_n| < \epsilon$, if n is sufficiently large. If τ_n , as approximating sum for ψ_n , is not uniquely determined, the conclusion holds for every τ_n having the approximating property. It was emphasized in an earlier paragraph that if T_n is an approximating function for f , there will exist a τ_n such that $\psi_n - \tau_n$ is identical with $f - T_n$. Consequently $|f - T_n| < \epsilon$, for all values of x , when n is sufficiently large; that is, T_n converges uniformly to the value of f .

The special hypothesis with regard to $E(y)$, and the conclusion, may be summarized as follows:

HYPOTHESIS F'. *Let $E(y)$ take on its minimum value * only for $y = 0$.*

THEOREM VIII. *If Hypotheses A, B'', and F' are satisfied; if $f(x)$ is a continuous function of period 2π , which can be represented for each positive integral value of n by a trigonometric sum of the n th order with an error nowhere exceeding ϵ_n in absolute value, so that $\lim_{n \rightarrow \infty} nE_2(\epsilon_n) = 0$, where E_2 has the meaning explained in the text; and if $T_n(x)$ is an approximating sum for $f(x)$; then $T_n(x)$ converges uniformly to the value of $f(x)$ as n becomes infinite.*

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* It has been pointed out that the assumption that the minimum value of $E(y)$ is zero, made for convenience in the course of the proof, does not affect the validity of the conclusion.

On a Rational Plane Quintic Curve with Four Real Cusps.

BY PETER FIELD.

Recent articles in the *Bulletin of the American Mathematical Society** recall the thought that the rational quintic curve with four cusps is of particular interest as it furnishes a simple illustration of a curve whose singular points can not be arbitrarily chosen. The rational sextic can not have its ten double points placed arbitrarily and has received some notice† and as the situation in the case of the quintic is so particularly simple it seems worthy of study.

A rational quintic has six singular points. If four of them are cusps whose position is given there will remain two points whose position is to be studied. It is well known‡ that $x : y : z$ the homogeneous coördinates of a point on a rational quintic curve may be written

$$x = \sum_{n=0}^{n=5} a_n \lambda^n, \quad y = \sum_{n=0}^{n=5} b_n \lambda^n, \quad z = \sum_{n=0}^{n=5} c_n \lambda^n, \quad (1)$$

where λ is an arbitrary parameter. Equations (1) can be simplified by taking the cuspidal points at $(1, 1, 1)$, $(1, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$ and taking the values of the parameter at these points as $0, 1, \infty$ and λ_1 . The equations then become

$$\begin{aligned} x &= (\lambda - \lambda_1)^2 ([\lambda_1 - 2\lambda_3 + 2\lambda_1\lambda_3]\lambda - \lambda_1\lambda_3), \\ y &= \lambda_1^3 (\lambda - 1)^2 (\lambda - \lambda_3), \\ z &= (\lambda - 1)^2 (\lambda - \lambda_1)^2 ([\lambda_1 - 2\lambda_3]\lambda - \lambda_1\lambda_3), \end{aligned} \quad (2)$$

λ_1 and λ_3 being arbitrary constants.

* Coolidge, "On the Existence of Curves with Assigned Singularities," Vol. 28, pp. 451-455 and Hollcroft, "Singularities of Curves of Given Order," Vol. 29, pp. 407-414.

† Valentiner, "Nogle Sætninger om visse algebraiske Kurver," *Tidsskrift for matematik*, Bd. V; Cayley, "A Second Memoir on Quartic Surfaces," *Proceedings of the London Mathematical Society*, Vol. III; Halphen, "Sur les courbes planes du sixième degré à neuf points doubles," *Bulletin de la Société Mathématique de France*, Vol. X.

‡ Riemann, "Theorie der Abelschen Funktionen," *Journ. f. Math.*, 1857 or text books on Higher Plane Curves such as Salmon or Clebsch.

The values of λ at points where the curve intersects the lines $z/x = k_1$ and $z/y = k_2$ are obtained from the equations *

$$\frac{(\lambda - 1)^2 ([\lambda_1 - 2\lambda_3]\lambda - \lambda_1\lambda_3)}{(\lambda_1 - 2\lambda_3 + 2\lambda_1\lambda_3)\lambda - \lambda_1\lambda_3} = k_1, \quad (3)$$

$$\frac{(\lambda - \lambda_1)^2 ([\lambda_1 - 2\lambda_3]\lambda - \lambda_1\lambda_3)}{\lambda_1^3(\lambda_1 - \lambda_3)} = k_2,$$

and at each of the double points k_1 and k_2 must have such values that equations (3) have two common roots. The conditions that k_1 and k_2 must satisfy are

$$\frac{\lambda_1^2(1 - k_2) - (1 - k_1)}{2(\lambda_1 - 1)} =$$

$$\frac{(2\lambda_1 - 4\lambda_3 + \lambda_1\lambda_3)\lambda_1^2(1 - k_2) - (2\lambda_1^2 - 3\lambda_1\lambda_3)(1 - k_1)}{-(\lambda_1 - 2\lambda_3 + 2\lambda_1\lambda_3)(1 - k_1) + \lambda_1^3(1 - k_2)} =$$

$$- \frac{(1 - k_1)(1 - k_2)2\lambda_1(1 - \lambda_1)}{\lambda_1^2(1 - k_2) - (1 - k_1)} \quad (4)$$

By replacing $1 - k_2$ and $1 - k_1$ by u and v these conditions may be written

$$(\lambda_1^2 u - v)^2 = 4\lambda_1(\lambda_1 - 1)^2 uv, \quad (5)$$

$$\lambda_1^5 u^2 - 2\lambda_1^2(\lambda_1 - \lambda_3 + \lambda_1\lambda_3)uv + (\lambda_1 - 2\lambda_3 + 2\lambda_1\lambda_3)v^2 -$$

$$2\lambda_1^2(\lambda_1 - 1)(2\lambda_1 - 4\lambda_3 + \lambda_1\lambda_3)u + 2\lambda_1(\lambda_1 - 1)(2\lambda_1 - 3\lambda_3)v = 0. \quad (6)$$

Regarded as equations in u and v the first equation represents a pair of lines through the origin and the second an hyperbola which also passes through the origin. The intersections at the origin determine the cuspidal point $(1 : 1 : 1)$ and the two remaining intersections give the values of u and v at the two unknown double points. Equation (5) gives

$$v = \lambda_1[2(\lambda_1 - 1)^2 + \lambda_1 \pm 2(\lambda_1 - 1)\sqrt{\lambda_1^2 - \lambda_1 + 1}]u, \quad (7)$$

and this substituted in (6) gives

$$u = \frac{\lambda_1 - 2\lambda_1^2 + \lambda_3(3\lambda_1 - 1) \mp (2\lambda_1 - 3\lambda_3)\sqrt{\lambda_1^2 - \lambda_1 + 1}}{2\lambda_1(\lambda_1 - 1)^2 + \lambda_1^2 + 4\lambda_3(\lambda_1 - 1)^3 + 3\lambda_3\lambda_1(\lambda_1 - 1) \pm [2\lambda_1(\lambda_1 - 1) + \lambda_3\{4(\lambda_1 - 1)^2 + \lambda_1\}]\sqrt{\lambda_1^2 - \lambda_1 + 1}} \quad (8)$$

From the form of these equations it appears that u and v are always real. Hence the

* For numerous references see Loria, "Spezielle ebene algebraische Kurven von höherer als der vierten Ordnung." *Encyklopädie der mathematischen Wissenschaften*, Bd. III, Heft 5; also Loria's book, *Spezielle algebraische und transcendente ebene Kurven*, p. 219.

THEOREM: *A rational plane quintic curve with four real cusps must have two additional real double points.*

The four cuspidal points and the six lines joining them in pairs, form a complete quadrangle which divides the plane into twelve triangles; it is sufficient to let the fifth double point be a point in one of these triangles. Suppose it is in region *A* (Fig. I). In that case u and v are opposite in sign and consequently from equation (5), λ_1 is negative. But if λ_1 is negative, equation (5) shows that the ratio v/u is negative for both double points and

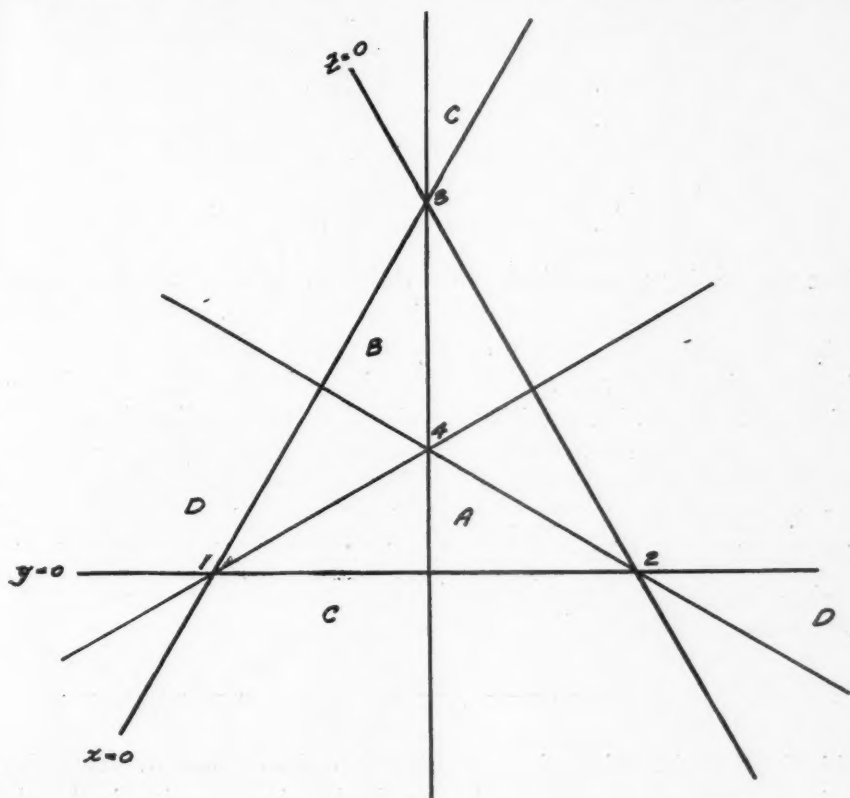


FIG. I.

the two double points are restricted to the triangles *A*, *B*, *C*, *D*. To generalize: the twelve triangles of the plane are divided into three sets of four; each set being composed of a triangle, such as *A*, and the three triangles which contain the angles vertical to the angles of the given triangle, such as *B*, *C*, and *D*: stated as a

THEOREM: *A rational plane quintic curve with four real cusps must have its two remaining singular points in the same set of triangles.**

The four cuspidal points 1, 2, 3, 4 (Fig. I) can be projected into themselves in twenty-four different ways. Of these there are eight which project A, B, C, D into themselves. They correspond to the substitutions in the group I.

Group I	Group II
1	1
23	$AB.CD$
14	$AD.BC$
23.14	$AC.BD$
12.34	AC
1342	$ADCB$
1243	$ABCD$
13.24	BD

the corresponding permutations of the regions A, B, C, D are given by the substitutions shown in group II which is the same group as I written so that it is simply isomorphic to itself. This shows that if there is a curve with singular points in A and B it can be projected into another with singular points in A and D , but not into one with the singular points in A and C . The values $\lambda_1 = -1$, $\lambda_3 = -2$ give $v = -3 \mp \frac{2}{3}\sqrt{3}$, $u = 29 \pm 5\frac{2}{3}\sqrt{3}$ and these values determine the coordinates of singular points in regions B and C . The substitution 23, i. e. projecting 1, 2, 3, 4 into 1, 3, 2, 4 projects the curve into one with singular points in A and D , while the substitution 12.34 transforms the curve into one with singular points in A and B . The values $\lambda_1 = -1$, $\lambda_3 = 2$ give

$$u = \frac{-167 \pm 98\sqrt{3}}{13}, \quad v = \frac{-7 \pm 18\sqrt{3}}{13}$$

This therefore determines singular points in regions D and B . The substitution 1243 transforms the curve into one having singular points in A and C . Hence the

* For figures showing the forms of rational quintic curves see Meyer, *Anwendungen der Topologie auf die Gestalten der Algebraischen Kurven vierter und fünfter Ordnung*, München (Diss.), 1878 also Field, "On the Forms of the Uncursal Quintic Curves," *American Journal of Mathematics*, Vol. XXVI. Some properties of the rational quintic curve with four cusps are given by Bassett, "On Quinquenodal and Sexnodal Quintics," *Quarterly Journal of Mathematics*, Vol. 37, pp. 208-209.

THEOREM: *The rational plane quintic curve with four real cusps can have its two remaining singular points in any two triangles of a given set of four.*

There remains the question: can the two singular points be in the same triangle? The values over which u and v range in each, of the given triangles are as follows:

region	A	B	C	D
u	0 to $-\infty$	0 to 1	1 to $+\infty$	0 to $-\infty$
v	0 to 1	0 to $-\infty$	0 to $-\infty$	1 to $+\infty$

For convenience it will be proved that the two singular points can not both lie in the region B , i. e. both values of u can not lie between 0 and 1. The product of the two values u_1 and u_2 of u given in equation (8) is

$$u_1 u_2 = \frac{-3\lambda_1^2 + \lambda_3(-2\lambda_1^2 + 10\lambda_1) + \lambda_3^2(3\lambda_1 - 8)}{-\lambda_1^3[\lambda_3^2 - 2(\lambda_1 - 1)\lambda_3 - \lambda_1]},$$

and as λ_1 is negative, this shows that if u_1 and u_2 correspond to points in B , λ_3 must also be negative. The equation

$$u_1 = \frac{\lambda_1 - 2\lambda_1^2 + \lambda_3(3\lambda_1 - 1) - (2\lambda_1 - 3\lambda_3)\sqrt{\lambda_1^2 - \lambda_1 + 1}}{2\lambda_1(\lambda_1 - 1)^2 + \lambda_1^2 + 4\lambda_3(\lambda_1 - 1)^3 + 3\lambda_3\lambda_1(\lambda_1 - 1) + [2\lambda_1(\lambda_1 - 1) + \lambda_3\{4(\lambda_1 - 1)^2 + \lambda_1\}]\sqrt{\lambda_1^2 - \lambda_1 + 1}}$$

represents an hyperbola with asymptotes parallel to the axes of reference, if u_1 and λ_3 are regarded as the variables. The hyperbola crosses the λ_3 axis at the point

$$\lambda_3 = \frac{\lambda_1^2 - 5\lambda_1 + \lambda_1\sqrt{\lambda_1^2 - \lambda_1 + 1}}{3\lambda_1 - 8},$$

and the vertical asymptote is at

$$\lambda_3 = \lambda_1 - 1 - \sqrt{\lambda_1^2 - \lambda_1 + 1}.$$

Similarly the equation

$$u_2 = \frac{\lambda_1 - 2\lambda_1^2 + \lambda_3(3\lambda_1 - 1) + (2\lambda_1 - 3\lambda_3)\sqrt{\lambda_1^2 - \lambda_1 + 1}}{2\lambda_1(\lambda_1 - 1)^2 + \lambda_1^2 + 4\lambda_3(\lambda_1 - 1)^3 + 3\lambda_3\lambda_1(\lambda_1 - 1) - [2\lambda_1(\lambda_1 - 1) + \lambda_3\{4(\lambda_1 - 1)^2 + \lambda_1\}]\sqrt{\lambda_1^2 - \lambda_1 + 1}}$$

represents an hyperbola which crosses the λ_3 axis at the point

$$\lambda_3 = \frac{\lambda_1^2 - 5\lambda_1 - \lambda_1\sqrt{\lambda_1^2 - \lambda_1 + 1}}{3\lambda_1 - 8}$$

and the vertical asymptote is at

$$\lambda_3 = \lambda_1 - 1 + \sqrt{\lambda_1^2 - \lambda_1 + 1}.$$

As much of the hyperbolas as is of interest is sketched in Fig. II. The relative position of the points E, F, G, H will be as shown independent of the parti-

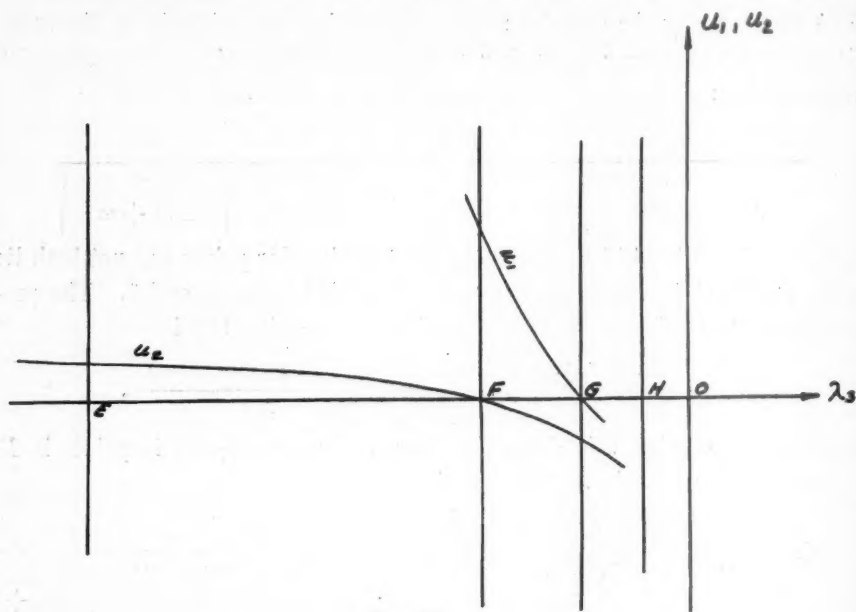


FIG. II.

cular negative value assigned to λ_1 . The only values of λ_3 which give positive values for both u_1 and u_2 lie between the points E and F , i. e. between

$$\lambda_3 = \lambda_1 - 1 - \sqrt{\lambda_1^2 - \lambda_1 + 1} \quad \text{and}$$

$$\lambda_3 = \frac{\lambda_1^2 - 5\lambda_1 - \lambda_1 \sqrt{\lambda_1^2 - \lambda_1 + 1}}{3\lambda_1 - 8}.$$

As u_1 decreases when λ_3 passes from the first value to the second, the theorem will be proved if the value of u_1 is greater than 1 at the point

$$\lambda_3 = \frac{\lambda_1^2 - 5\lambda_1 - \lambda_1 \sqrt{\lambda_1^2 - \lambda_1 + 1}}{3\lambda_1 - 8}.$$

The value of u_1 at this point is

$$u_1 = \frac{-3\lambda_1^3 + 11\lambda_1^2 - 11\lambda_1 + 8 + (3\lambda_1^2 - 11\lambda_1 + 8) \sqrt{\lambda_1^2 - \lambda_1 + 1}}{-3\lambda_1^3 + 8\lambda_1^2}$$

which is greater than one for all negative values of λ_1 . Hence the final

THEOREM: *A rational plane quintic curve with four real cusps has two additional real singular points which must lie in two different triangles of the same set.*

UNIVERSITY OF MICHIGAN,
March, 1924.

Projective Properties of a Ruled Surface in the Neighborhood of a Ruling.*

By A. F. CARPENTER.

I. INTRODUCTION.

On every ruled surface S (not a quadric) there lie two curves, the flecnodal curve C_F and the complex curve C_C , each of which cuts each ruling g of the surface in two points. The curves are in general distinct, and will be so taken here. These four points on each ruling constitute a harmonic set.

The planes which osculate C_F at the points P_y, P_z , in which it cuts g intersect in a line $l_{\phi\psi}$. There corresponds to this line by means of the linear complex L which osculates S along g a second line $l_{a\beta}$, the polar reciprocal of $l_{\phi\psi}$ with respect to L . The three lines $g, l_{\phi\psi}, l_{a\beta}$, are in general non-intersecting and so determine a quadric Q_1 .

The quadric Q_1 and the quadric Q osculating S along g intersect in a space quartic, but since g is common to both Q_1 and Q , and since they are not tangent along g the remaining part of this curve is a space cubic $C_{F'}$, called the *primary flecnodal cubic*.

Let us think of the plane which osculates $C_{F'}$ at one of its points P_ξ . There corresponds to this plane by means of L a point P_η lying in it. The locus of P_η for all the osculating planes of $C_{F'}$ proves to be a second space cubic $C_{F''}$ called the *secondary flecnodal cubic*. The points of $C_{F'}$ and $C_{F''}$ are thus put into (1, 1) correspondence and by means of this correspondence a ruled surface T is determined, the rulings of T being lines joining corresponding pairs of points of $C_{F'}$ and $C_{F''}$.

The planes which osculate C_C at the points P_y, P_z in which it cuts g intersect in a line $l_{\phi\bar{\psi}}$. There corresponds to this line by means of L a second line $l_{a\bar{\beta}}$, the polar reciprocal of $l_{\phi\bar{\psi}}$ with respect to L . The three lines $g, l_{\phi\bar{\psi}}, l_{a\bar{\beta}}$, determine a quadric Q_2 which intersects Q in g and in a space cubic $C_{C'}$ called the *primary complex cubic*. By a process identical with that used above but involving $C_{C'}$ and L , we define the *secondary complex cubic*, $C_{C''}$. The (1, 1) point correspondence between $C_{C'}$ and $C_{C''}$ determines as before a ruled surface \bar{T} .

* Presented to the American Mathematical Society, San Francisco Section, September, 1923.

The curves $C_{F'}$ and $C_{C'}$ both lie upon Q . We may therefore put the points of these two curves into correspondence by making to correspond those points which lie upon the same ruling of Q . But since there are two reguli upon Q it follows that $C_{F'}$ and $C_{C'}$ can thus be put into correspondence in either of two ways. We shall consider here the correspondence between $C_{F'}$ and $C_{C'}$ which is determined by that regulus of Q made up of the lines which intersect g . Having set up point correspondences between the pairs of curves $(C_{F'}, C_{F''})$, $(C_{C'}, C_{C''})$, $(C_{F'}, C_{C'})$, there is thereby determined a point correspondence between $C_{F''}$ and $C_{C''}$ and a ruled surface R made up of lines joining the corresponding points of $C_{F''}$ and $C_{C''}$.

In a paper to appear in the *Tohoku Mathematical Journal* * the above curves and surfaces have been discussed. We introduce here without further reference certain equations and theorems contained in this paper.

The equations of the two quadrics Q_1, Q_2 are respectively

$$(Q_1) \quad p_{12}^2 p_{21} x_1 x_3 - p_{12} p_{21}^2 x_2 x_4 - 2p_{12}^2 q_{21} x_2^2 + 2p_{21}^2 q_{12} x_2^2 = 0,$$

$$(Q) \quad P(p_{12}^2 p_{21} x_1 x_3 - p_{12} p_{21}^2 x_2 x_4 - 2p_{12}^2 q_{21} x_2^2 + 2p_{21}^2 q_{12} x_2^2) \\ - p_{12} p_{21} D(p_{12} x_2^2 - p_{21} x_4^2) = 0.$$

The parametric equations of $C_{F'}$, $C_{F''}$, $C_{C'}$, $C_{C''}$ are

$$(C_{F'}) \quad \xi_1 = 2t(p_{12}^2 q_{21} t^2 - p_{21}^2 q_{12}), \quad \xi_2 = 2(p_{12}^2 q_{21} t^2 - p_{21}^2 q_{12}),$$

$$\xi_3 = p_{12} p_{21} t(p_{12} t^2 - p_{21}), \quad \xi_4 = p_{12} p_{21} (p_{12} t^2 - p_{21});$$

$$(C_{F''}) \quad \eta_1 = 2p_{21}(3p_{12}^2 q_{21} t^2 + p_{21}^2 q_{12}), \quad \eta_2 = 2p_{12} t(p_{12}^2 q_{21} t^2 + 3p_{21}^2 q_{12}),$$

$$\eta_3 = p_{12} p_{21}^2 (3p_{12} t^2 + p_{21}), \quad \eta_4 = p_{12}^2 p_{21} t(p_{12} t^2 + 3p_{21});$$

$$\bar{\xi}_1 = 2Pt(p_{12}^2 q_{21} t^2 - p_{21}^2 q_{12}) + p_{12} p_{21} Dt(p_{12} t^2 - p_{21}) = P\xi_1 + D\xi_3,$$

$$(C_{C'}) \quad \bar{\xi}_2 = 2P(p_{12}^2 q_{21} t^2 - p_{21}^2 q_{12}) + p_{12} p_{21} D(p_{12} t^2 - p_{21}) = P\xi_2 + D\xi_4,$$

$$\bar{\xi}_3 = p_{12} p_{21} Pt(p_{12} t^2 - p_{21}) = P\xi_3, \quad \bar{\xi}_4 = p_{12} p_{21} P(p_{12} t^2 - p_{21}) = P\xi_4;$$

$$\bar{\eta}_1 = 2p_{21} P(3p_{12}^2 q_{21} t^2 + p_{21}^2 q_{12}) + p_{12} p_{21}^2 D(3p_{12} t^2 + p_{21}) = P\eta_1 + D\eta_3,$$

$$(C_{C''}) \quad \bar{\eta}_2 = 2p_{12} Pt(p_{12}^2 q_{21} t^2 + 3p_{21}^2 q_{12}) + p_{12}^2 p_{21} Dt(p_{12} t^2 + 3p_{21}) = P\eta_2 + D\eta_4,$$

$$\bar{\eta}_3 = p_{12} p_{21}^2 P(3p_{12} t^2 + p_{21}) = P\eta_3, \quad \bar{\eta}_4 = p_{12}^2 p_{21} Pt(p_{12} t^2 + 3p_{21}) = P\eta_4;$$

where

$$(1) \quad P = p_{12} q_{21} - p_{21} q_{12}, \quad D = \frac{p_{21}}{p_{12}} \Delta_1 - \frac{p_{12}}{p_{21}} \Delta_2,$$

$$\Delta_1 = p_{12}^2 q_{22} - p_{12} q'_{12} + 3q_{12}^2, \quad \Delta_2 = p_{21}^2 q_{11} - p_{21} q'_{21} + 3q_{21}^2.$$

* Accepted September, 1922.

and where p_{ik}, q_{ik} are coefficients of the system of differential equations

$$(F) \quad \begin{aligned} y'' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + q_{21}y + q_{22}z &= 0, \end{aligned}$$

defining the surface S when referred to the two branches of C_F as its directrix curves.* p_{ik}, q_{ik} are functions of an independent variable x .

The equations of the planes osculating $C_{F'}, C_{F''}, C_{O'}, C_{O''}$ at points given by the parameter t are, respectively

$$(2) \quad \begin{aligned} p_{12}p_{21}(3p_{12}t^2 + p_{21})x_1 - p_{12}p_{21}t(p_{12}t^2 + 3p_{21})x_2 \\ - 2(3p_{12}^2q_{21}t^2 + p_{21}^2q_{12})x_3 + 2t(p_{12}^2q_{21}t^2 + 3p_{21}^2q_{12})x_4 = 0; \end{aligned}$$

$$(3) \quad \begin{aligned} p_{12}^2p_{21}t(p_{12}t^2 - p_{21})x_1 - p_{12}p_{21}^2(p_{12}t^2 - p_{21})x_2 \\ - 2p_{12}t(p_{12}^2q_{21}t^2 - p_{21}^2q_{12})x_3 \\ + 2p_{21}(p_{12}^2q_{21}t^2 - p_{21}^2q_{12})x_4 = 0; \end{aligned}$$

$$(4) \quad \begin{aligned} P[p_{12}p_{21}(3p_{12}t^2 + p_{21})x_1 - p_{12}p_{21}t(p_{12}t^2 + 3p_{21})x_2 \\ - 2(3p_{12}^2q_{21}t^2 + p_{21}^2q_{12})x_3 + 2t(p_{12}^2q_{21}t^2 + 3p_{21}^2q_{12})x_4] \\ - p_{12}p_{21}D[(3p_{12}t^2 + p_{21})x_3 - t(p_{12}t^2 + 3p_{21})x_4] = 0; \end{aligned}$$

$$(5) \quad \begin{aligned} P[p_{12}^2p_{21}t(p_{12}t^2 - p_{21})x_1 - p_{12}p_{21}^2(p_{12}t^2 - p_{21})x_2 \\ - 2p_{12}t(p_{12}^2q_{21}t^2 - p_{21}^2q_{12})x_3 + 2p_{21}(p_{12}^2q_{21}t^2 - p_{21}^2q_{12})x_4] \\ - p_{12}p_{21}D[p_{12}t(p_{12}t^2 - p_{21})x_3 - p_{21}(p_{12}t^2 - p_{21})x_4] = 0. \end{aligned}$$

The quadric Q_1 , the cubics $C_{F'}$, $C_{F''}$, and the surface T are respectively projectively equivalent to the quadric Q_2 , the cubics $C_{O'}$, $C_{O''}$ and the surface \bar{T} , the first four being projected into the second four by one and the same transformation

$$(6) \quad x_1 = P\bar{x}_1 - D\bar{x}_3, \quad x_2 = P\bar{x}_2 - D\bar{x}_4, \quad x_3 = P\bar{x}_3, \quad x_4 = P\bar{x}_4.$$

The system of differential equations defining the ruled surface T when referred to $C_{F'}$ and $C_{F''}$ as directrix curves, is

$$(T) \quad \begin{aligned} \xi'' + \pi_{11}\xi' + x_{11}\xi + x_{12}\eta &= 0, \\ \eta'' + \pi_{11}\eta' + x_{21}\xi + x_{11}\eta &= 0, \end{aligned}$$

where

$$\begin{aligned} \pi_{11} &= -(3p_{12}t^2 - p_{21})\theta, \quad x_{11} = 3p_{12}t\theta, \quad x_{12} = \theta, \\ x_{21} &= 3p_{12}p_{21}\theta, \quad \theta = \frac{1}{t(p_{12}t^2 - p_{21})}, \end{aligned}$$

and where differentiation is with respect to t .

* Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, p. 149 et seq. This text hereafter referred to as *Proj. Dif. Geom.*

C_F and C_F'' are asymptotic curves on T . T belongs to a linear congruence with distinct directrices and these two straight lines are the two branches of the flecnodal curve of T . Referred to these two straight lines as directrix curves, the system of differential equations defining T is

$$(T'') \quad \begin{aligned} \xi'' + \pi_{11}\xi' + (x_{11} + \mu x_{12})\xi &= 0, \\ \eta'' + \pi_{11}\eta' + (x_{11} - \mu x_{12})\eta &= 0, \end{aligned}$$

where $\mu^2 = 3p_{12}p_{21}$.

Since \bar{T} is projectively equivalent to T , all the properties just assigned to T belong also to \bar{T} , and equations (T) , (T'') , serve equally well for the system defining \bar{T} .

The ruled surface R , when referred to C_F'' , C_O'' as directrix curves, has for its defining system of differential equations

$$(R) \quad \begin{aligned} \eta'' + P_{11}\eta' + P_{12}\bar{\eta}' + Q_{11}\eta + Q_{12}\bar{\eta} &= 0, \\ \bar{\eta}'' + P_{21}\eta' + P_{22}\bar{\eta}' + Q_{21}\eta + Q_{22}\bar{\eta} &= 0, \end{aligned}$$

where

$$\begin{aligned} DP_{11} &= -8P^2t^3\theta^2 - 4p_{12}Dt^2\theta, & DP_{12} &= 8Pt^3\theta^2, \\ DP_{21} &= -8P^3t^3\theta^2, & DP_{22} &= 8P^2t^3\theta^2 - 4p_{12}Dt^2\theta, \\ DQ_{11} &= 12P^2t^2\theta^2 + 6p_{12}Dt\theta, & DQ_{12} &= -12Pt^2\theta^2, \\ DQ_{21} &= 12P^3t^2\theta^2, & DQ_{22} &= -12P^2t^2\theta^2 + 6p_{12}Dt\theta. \end{aligned}$$

R belongs to a linear congruence with coincident directrices. These lines coincide in the ruling g of S and constitute the complete flecnodal curve of R .

The three ruled surfaces T , \bar{T} , R have each two and only two lines in common with Q , those common to R and Q being the flecnodal tangents to S at P_y , P_z .

The tetrahedron of reference for which the equations above hold true has for its vertices the two points P_y , P_z , and the two points P_ρ , P_σ , located on the respective flecnodal tangents to S at P_y , P_z . The values of ρ , σ are

$$\rho = 2y' + p_{12}z, \quad \sigma = 2z' + p_{21}y.$$

In this system of coordinates the equation of the osculating quadric Q in point coordinates is

$$(Q) \quad x_1x_4 - x_2x_3 = 0,$$

and that of the osculating linear complex L , in line coordinates, is

$$(L) \quad p_{12}\omega_{13} + p_{21}\omega_{42} = 0.*$$

* *Proj. Dif. Geom.*, pp. 191, 206.

By a suitable choice of the unit point we associate with each point P_x of space a linear function of y, z, ρ, σ ,

$$(P) \quad x_k = \alpha y_k + \beta z_k + \gamma \rho_k + \delta \sigma_k \quad (k=1, 2, 3, 4),$$

whose coefficients are coördinates of P_x ; ($x_1 = \alpha, x_2 = \beta, x_3 = \gamma, x_4 = \delta$). This implies of course that $y_1 = 1, y_2 = y_3 = y_4 = 0$; $z_2 = 1, z_1 = z_3 = z_4 = 0$, etc. The subscript k will ordinarily be omitted.*

II. THE ASSOCIATED LINEAR COMPLEXES.

The planes osculating any space cubic at the points in which it is cut by any plane

$$(7) \quad u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0,$$

intersect in a point which lies in this plane. There is thus determined by each cubic a null-system in which to each plane of space there corresponds a point lying in that plane. All the lines of space which are such that to each of their points there corresponds by such a null-system a plane containing the line, constitute a linear complex. We proceed to find the point-plane correspondence and the linear complex determined by the primary flecnodal cubic.

If we substitute from (C_F) in (7) we have for the determination of the values of t corresponding to the three points of intersection, the equation

$$(8) \quad (2p_{12}^2 q_{21} u_1 + p_{12}^2 p_{21} u_3) t^3 + (2p_{12}^2 q_{21} u_2 + p_{12}^2 p_{21} u_4) t^2 - (2p_{21}^2 q_{12} u_1 + p_{12} p_{21}^2 u_3) t - (2p_{21}^2 q_{12} u_2 + p_{12} p_{21}^2 u_4) = 0.$$

From (8) we have

$$(9) \quad \frac{2q_{21}u_2 + p_{21}u_4}{2q_{21}u_1 + p_{21}u_3} = -(t_1 + t_2 + t_3) = -A, \quad \frac{p_{21}^2(2q_{12}u_1 + p_{12}u_3)}{p_{12}^2(2q_{21}u_1 + p_{21}u_3)} = -(t_1 t_2 + t_2 t_3 + t_3 t_1) = -B, \quad \frac{p_{21}^2(2q_{12}u_2 + p_{12}u_4)}{p_{12}^2(2q_{21}u_1 + p_{21}u_3)} = t_1 t_2 t_3 = C.$$

From (9) we find for the coördinates of the plane (7) the expressions

$$(10) \quad \begin{aligned} u_1 &= -p_{12}p_{21}(p_{12}B + p_{21}), & u_2 &= p_{12}p_{21}(p_{21}A + p_{12}C), \\ u_3 &= 2(p_{12}^2 q_{21}B + p_{21}^2 q_{12}), & u_4 &= -2(p_{21}^2 q_{12}A + p_{12}^2 q_{21}C). \end{aligned}$$

Introducing in (2) for t the values t_1, t_2, t_3 in turn, we can write down from the resulting three equations quantities proportional to the coördinates of the point of intersection of these three osculating planes. They are

* See *Proj. Dif. Geom.*, pp. 127, 129, 191.

$$(11) \quad \begin{aligned} x_1 &= 2(-p_{21}^2 q_{12} A + 3p_{12}^2 q_{21} C), & x_2 &= 2(p_{12}^2 q_{21} B - 3p_{21}^2 q_{12}), \\ x_3 &= p_{12} p_{21} (-p_{21} A + 3p_{12} C), & x_4 &= p_{12} p_{21} (p_{12} B - 3p_{21}). \end{aligned}$$

The plane (10) and the point (11) correspond to each other. To get the analytic expressions for this correspondence it is only necessary to eliminate A, B, C from (10) and (11). We find for the coördinates of the plane in terms of those of the point, the expressions

$$(12) \quad \begin{aligned} u_1 &= 2p_{12} p_{21} x_2 - (13)x_4, & u_2 &= -2p_{12} p_{21} x_1 + (31)x_3, \\ u_3 &= -(31)x_2 + 8q_{12} q_{21} x_4, & u_4 &= (13)x_1 - 8q_{12} q_{21} x_3, \end{aligned}$$

where

$$(13) = p_{12} q_{21} + 3p_{21} q_{12}, \quad (31) = 3p_{12} q_{21} + p_{21} q_{12}.$$

It will be convenient to write this correspondence in inverse form. It proves to be

$$(12') \quad \begin{aligned} x_1 &= -8q_{12} q_{21} u_2 - (31)u_4, & x_2 &= 8q_{12} q_{21} u_1 + (13)u_3, \\ x_3 &= -(13)u_2 - 2p_{12} p_{21} u_4, & x_4 &= (31)u_1 + 2p_{12} p_{21} u_3. \end{aligned}$$

The equation in line coördinates of the linear complex of which (12) or (12') is the null-system, can be obtained as follows. Let $P; (x_1, \dots, x_4)$ and $P'; (x'_1, \dots, x'_4)$ be two points of a line of this complex. If (u_1, \dots, u_4) are the coördinates of the plane corresponding to P , then must P' as well as P be in this plane, so that

$$u_1 x'_1 + u_2 x'_2 + u_3 x'_3 + u_4 x'_4 = 0.$$

Introduce into the above equation the values of u_1, \dots, u_4 from (12) and we find for the equation of the complex

$$(13) \quad 2p_{12} p_{21} \omega_{12} - (13)\omega_{14} + (31)\omega_{23} + 8q_{12} q_{21} \omega_{34} = 0,$$

where

$$\omega_{ij} = x_i x'_j - x_j x'_i,$$

If we think of a line of this complex as being determined by two planes rather than by two lines, we find for its equation by a process similar to the above,

$$(13') \quad 8q_{12} q_{21} \omega'_{12} + (31)\omega'_{14} - (13)\omega'_{23} + 2p_{12} p_{21} \omega'_{34} = 0,$$

where

$$\omega'_{ij} = u_i u'_j - u_j u'_i.$$

We shall call this complex the *first associated complex* and designate it by the symbol L_1 .

The linear complex determined by the secondary flecnodal cubic is obtained by applying the above process to equations (C_F'') and (3). Four quan-

titles proportional to the coördinates of the plane determined by the three points t_1, t_2, t_3 of C_F'' are

$$(14) \quad \begin{aligned} v_1 &= p_{12}^2 p_{21} (p_{21} A - 3 p_{12} C), & v_2 &= p_{12} p_{21} (p_{12} B - 3 p_{21}), \\ v_3 &= -2 p_{12} (p_{21}^2 q_{12} A - 3 p_{12}^2 q_{21} C), & v_4 &= -2 p_{21} (p_{12}^2 q_{21} B - 3 p_{21}^2 q_{12}). \end{aligned}$$

For the point of intersection of the three osculating planes at these points we find the coördinates

$$(15) \quad \begin{aligned} x_1 &= 2 p_{21} (p_{12}^2 q_{21} B + p_{21}^2 q_{12}), & x_2 &= 2 p_{12} (p_{21}^2 q_{12} A + p_{12}^2 q_{21} C), \\ x_3 &= p_{12} p_{21}^2 (p_{12} B + p_{21}), & x_4 &= p_{12}^2 p_{21} (p_{21} A + p_{12} C). \end{aligned}$$

The elimination of A, B, C from (14) and (15) gives us the analytic expressions for the point-plane correspondence determined by C_F'' . They are precisely those given in (12) (or (12')), so that we may state the

THEOREM: *The primary and secondary flecnodal cubics associated with each ruling g of a ruled surface S determine the same linear complex.*

By a process identical with that employed in obtaining the null-system and the equation of the first associated complex we can obtain from equations (C_C') and (C_C'') of the primary and secondary complex cubics, and from the equations (4) and (5) of their osculating planes, analytic expressions for the point-plane correspondence determined by these curves and the equation of the resulting linear complex. But these results can be obtained in a much simpler way if we remember that the curves C_C', C_C'' are respectively projectively equivalent to the curves C_F', C_F'' .

Transformation (6) expressed in plane coördinates is

$$(16) \quad u_1 = P\bar{u}_1, \quad u_2 = P\bar{u}_2, \quad u_3 = P\bar{u}_3 + D\bar{u}_1, \quad u_4 = P\bar{u}_4 + D\bar{u}_1,$$

and in line coördinates is

$$(17) \quad \begin{aligned} \omega_{12} &= P^2 \bar{\omega}_{12} - PD(\bar{\omega}_{14} - \bar{\omega}_{23}) + D^2 \bar{\omega}_{34}, \\ \omega_{13} &= P^2 \bar{\omega}_{13}, \quad \omega_{14} = P^2 \bar{\omega}_{14} - PD \bar{\omega}_{34}, \quad \omega_{23} = P^2 \bar{\omega}_{23} + PD \bar{\omega}_{34}, \\ \omega_{42} &= P^2 \bar{\omega}_{42}, \quad \omega_{34} = P^2 \bar{\omega}_{34}. \end{aligned}$$

In order to obtain the analytic expressions for the point-plane correspondence determined by C_C' and C_C'' it will be sufficient to apply transformations (6) and (16) to (12). There result the equations

$$(18) \quad \begin{aligned} u_1 &= 2 p_{12} p_{21} P^2 x_2 - P[2 p_{12} p_{21} D + (13)P] x_4, \\ u_2 &= -2 p_{12} p_{21} P^2 x_1 + P[2 p_{12} p_{21} D + (31)P] x_3, \\ u_3 &= -P[2 p_{12} p_{21} D + (31)P] x_2 + 2(p_{12} D + 2 q_{12} P)(p_{21} D + 2 q_{21} P) x_4, \\ u_4 &= P[2 p_{12} p_{21} D + (13)P] x_1 - 2(p_{12} D + 2 q_{12} P)(p_{21} D + 2 q_{21} P) x_3. \end{aligned}$$

Similarly we obtain the equation of the linear complex determined by C_C' and C_C'' by applying transformation (17) to equation (13). There results

$$(19) \quad P^2[2p_{12}p_{21}\omega_{12} - (13)\omega_{14} + (31)\omega_{23} + 8q_{12}q_{21}\omega_{34}] - 2PDp_{12}p_{21}(\omega_{14} - \omega_{23}) + \{PD[(13) + (31) + 2p_{12}p_{21}D^2]\omega_{34} = 0.$$

We shall call this linear complex the *second associated complex* and designate it by the symbol L_2 . Very obviously we might state a new theorem for the complex cubics obtained from that for the flecnodal cubics by the interchange of the words "flecnodal" and "complex."

III. PROPERTIES OF L_1 AND L_2 . THE ASSOCIATED CONGRUENCES.

It has been noted that the ruled surface T belongs to a linear congruence with distinct directrices. We shall determine this congruence and its directrices.

The point-plane correspondence determined by L is given by the equations

$$(20)^* \quad u_1 = p_{12}x_3, \quad u_2 = -p_{21}x_4, \quad u_3 = -p_{12}x_1, \quad u_4 = p_{21}x_2.$$

If we make use of (20) and (3) to find the point corresponding to the plane osculating C_F'' at any point P_η , there results precisely the four coordinates given by (C_F') . It follows that each of the planes osculating C_F' and C_F'' at a pair of corresponding points P_ξ, P_η , contains both these points and hence that their line of intersection is the line joining these points. This line is therefore self-polar with respect to L , i. e. it is a line of L . But $l_{\xi\eta}$ is the general ruling of T so that T belongs to L .

The point which corresponds to the plane osculating C_F' at P_ξ by means of L_1 is of course P_ξ itself and the point which corresponds to the plane osculating C_F'' at P_η by means of L_1 is P_η since L_1 bears the same relation to C_F'' that it does to C_F' . It follows that $l_{\xi\eta}$ is a line of L_1 . Combining these results we can enunciate the

THEOREM: *The ruled surface T determined by the point-to-point correspondence between the primary and secondary flecnodal cubics associated with each ruling g of S , belongs to that linear congruence common to the first associated linear complex and the linear complex osculating S along g .*

We shall call this congruence the *first associated congruence* and denote it by the symbol Γ_1 .

* *Proj. Dif. Geom.*, p. 206.

From equations (L) and (17) it follows that the complex L is invariant under the projective transformation which carries L_1 into L_2 and hence that the linear congruence Γ_2 common to L and L_2 is projectively equivalent to the congruence Γ_1 and that \bar{T} belongs to Γ_2 . We shall speak of Γ_2 as the *second associated congruence*.

The two directrices of a linear congruence thought of as the intersection of two linear complexes, are the locus of all the points of space for which the planes corresponding to them by means of the two complexes coincide. We shall use this property to determine the directrices of the congruence Γ_1 .

The planes corresponding to any point P_x by means of L_1 and L will, by (12) and (20), coincide providing

$$(21) \quad \begin{aligned} \dots 2p_{12}p_{21}x_2 - kp_{12}x_3 - (13)x_4 &= 0, \\ 2p_{12}p_{21}x_1 \dots - (31)x_3 - kp_{21}x_4 &= 0, \\ kp_{12}x_1 - (31)x_2 \dots + 8q_{12}q_{21}x_4 &= 0, \\ (13)x_1 - kp_{21}x_2 - 8q_{12}q_{21}x_3 \dots &= 0, \end{aligned}$$

where k is an arbitrary function. The parametric equations of this locus will be obtained by solving system (21) for x_1, \dots, x_4 . This is possible if, and only if, the determinant of the coefficients vanishes identically, that is, if

$$(22) \quad (p_{12}p_{21}k^2 - 3P^2)^2 \equiv 0,$$

The two roots of (22) are $k_1 = \frac{\sqrt{3}P}{\sqrt{p_{12}p_{21}}}$ and $k_2 = -\frac{\sqrt{3}P}{\sqrt{p_{12}p_{21}}}$ and for either of these values of k equations (21) are consistent. Since in each case the matrix of the coefficients of (21) is of rank two, the four equations are those of four planes of a pencil. We may therefore use any pair of them to define our directrices, there being two such lines, one for each value of k . We shall use the first pair.

It will be useful to determine two points upon each of these two lines. For this purpose we seek to find their points of intersection with Q . Solving the first pair of equations (21) simultaneously with equation (Q) we find, when $k = k_1$, the two points given by the expressions

$$(23) \quad \begin{aligned} \hat{u} &= -2\sqrt{p_{21}q_{12}}y + 2\sqrt{3p_{12}q_{12}z} - p_{12}\sqrt{p_{21}}\rho + p_{12}\sqrt{3p_{12}}\sigma, \\ \hat{r} &= 2\sqrt{3p_{21}q_{21}}y + 2\sqrt{p_{12}q_{21}z} + p_{21}\sqrt{3p_{21}}\rho + p_{21}\sqrt{p_{12}}\sigma, \end{aligned}$$

and when $k = k_2$, the two points whose expressions are

$$(24) \quad \begin{aligned} \hat{v} &= 2\sqrt{p_{21}q_{12}}y + 2\sqrt{3p_{12}q_{12}z} + p_{12}\sqrt{p_{21}}\rho + p_{12}\sqrt{3p_{12}}\sigma, \\ \hat{s} &= -2\sqrt{3p_{21}q_{21}}y + 2\sqrt{p_{12}q_{21}z} - p_{21}\sqrt{3p_{21}}\rho + p_{21}\sqrt{p_{12}}\sigma. \end{aligned}$$

The general element of the osculating quadric Q of the same kind as g , cuts the flecnodal tangents $l_{y\rho}$, $l_{z\sigma}$, in two points given by the respective expressions $ay + \beta\rho$, $az + \beta\sigma$. The equations of the planes corresponding to these two points in the null-system determined by L_1 are, by (12)

$$(25) \quad \begin{aligned} [-2p_{12}p_{21}a + (31)\beta]x_2 + [(13)a - 8q_{12}q_{21}\beta]x_4 &= 0, \\ [2p_{12}p_{21}a - (13)\beta]x_1 + [-(31)a + 8q_{12}q_{21}\beta]x_3 &= 0. \end{aligned}$$

If the line joining these two points is to belong to L_1 then must it coincide with the line of intersection of the two planes (12). The general point on the line determined by the points $ay + \beta\rho$, $az + \beta\sigma$, has coordinates proportional to $(\kappa a, \lambda a, \kappa\beta, \lambda\beta)$ and these coordinates will satisfy equations (12) if and only if

$$p_{12}p_{21}a^2 - 2(p_{12}q_{21} + p_{21}q_{12})a\beta + 4q_{12}q_{21}\beta^2 = 0,$$

that is, if

$$a/\beta = 2q_{12}/p_{12} \quad \text{or} \quad a/\beta = 2q_{21}/p_{21}.$$

There are therefore two and only two, rulings of Q of the same kind as g which belong to L_1 . One of these ruling cuts $l_{y\rho}$, $l_{z\sigma}$, in the respective points P_u , P_v , where

$$(26) \quad u = 2q_{12}y + p_{12}\rho, \quad v = 2q_{12}z + p_{12}\sigma,$$

and the other cuts $l_{y\rho}$, $l_{z\sigma}$, in the respective points P_r , P_s where

$$(27) \quad r = 2q_{21}y + p_{21}\rho, \quad s = 2q_{21}z + p_{21}\sigma.$$

Comparison of (23), (24), (26), (27), shows that

$$(28) \quad \begin{aligned} \hat{u} &= -\sqrt{p_{21}}u + \sqrt{3p_{12}}v, & \hat{r} &= \sqrt{3p_{21}}r + \sqrt{p_{12}}s, \\ \hat{v} &= \sqrt{p_{21}}u + \sqrt{3p_{12}}v, & \hat{s} &= -\sqrt{3p_{21}}r + \sqrt{p_{12}}s, \end{aligned}$$

so that the two lines which Q and L_1 have in common intersect the two directrices of Γ_1 . Moreover the four points P_u , $P_{\hat{u}}$, P_v , $P_{\hat{v}}$ as well as the four points P_r , P_r , P_s , $P_{\hat{s}}$ constitute harmonic ranges. Combining the foregoing results we can state the

THEOREM: *The osculating quadric Q and the first associated linear congruence Γ_1 have two and only two lines in common and for each of these lines the points in which it is cut by the two flecnodal tangents are harmonically separated by the points in which it is cut by the two directrices of Γ_1 .*

These two lines common to Γ_1 and Q are of course the two which T and Q have in common. They play an important part in defining the principle

surface of the flecnode congruence.* In their totality they constitute two ruled surface of the flecnode congruence. Their defining systems of differential equations are readily found. They are, for the surface generated by l_{uv} ,

$$\begin{aligned} u'' - \left(\frac{\pi'}{\pi} + \frac{2q_{12}}{p_{12}} \right) u' + \frac{p_{12}}{2\pi} (\bar{\pi} + \pi) v' + \\ \left(\frac{1}{4} p_{12} p_{21} \frac{\pi}{\pi} - \frac{\pi}{4p_{12}} + 3 \frac{q_{12} \pi'}{p_{12} \pi} - 3 \frac{p_{12} q'_{12} - q^2_{12}}{p^2_{12}} \right) u \\ + \left[\frac{3q_{12}}{2\pi} (\pi - \bar{\pi}) - \frac{1}{2} p_{12} \frac{\pi'}{\pi} \right] v = 0, \end{aligned} \quad (29)$$

$$\begin{aligned} v'' + \frac{p_{21}}{2\pi} (\bar{\pi} + \pi) u' - \left(\frac{\pi'}{\pi} + \frac{2q_{12}}{p_{12}} \right) v' + \left[\frac{3p_{21} q_{12}}{2p_{12} \pi} (\bar{\pi} - \pi) - \frac{1}{2} p_{21} \frac{\pi'}{\pi} + \frac{P}{p_{12}} \right] u \\ + \left(\frac{1}{4} p_{12} p_{21} \frac{\pi}{\pi} - \frac{\pi}{4p_{12}} + 3 \frac{q_{12} \pi'}{p_{12} \pi} - 3 \frac{p_{12} q'_{12} - q^2_{12}}{p^2_{12}} \right) v = 0; \end{aligned}$$

and for that generated by l_{rs} ,

$$\begin{aligned} r'' - \left(\frac{x'}{\bar{x}} + \frac{2q_{21}}{p_{21}} \right) r' + \frac{p_{12}}{2x} (\bar{x} + x) s' \\ + \left[\frac{1}{4} p_{12} p_{21} \frac{\bar{x}}{x} - \frac{\bar{x}}{4p_{21}} + \frac{3q_{21} x'}{p_{21} \bar{x}} - 3 \frac{p_{21} q'_{21} - q^2_{21}}{p^2_{21}} \right] r \\ + \left[\frac{3p_{12} q_{21}}{2p_{21} x} (x - \bar{x}) - \frac{1}{2} p_{12} \frac{x'}{\bar{x}} - \frac{P}{p_{21}} \right] s = 0, \\ (30) \quad s'' + \frac{p_{21}}{2x} (x + \bar{x}) r' - \left(\frac{x'}{\bar{x}} + \frac{2q_{21}}{p_{21}} \right) s' + \left[\frac{3q_{21}}{2\bar{x}} (\bar{x} - x) - \frac{1}{2} p_{21} \frac{x'}{\bar{x}} \right] r \\ + \left[\frac{1}{4} p_{12} p_{21} \frac{x}{\bar{x}} + 3 \frac{q_{21} x'}{p_{21} x} - \frac{x}{4p_{21}} - 3 \frac{p_{21} q'_{21} - q^2_{21}}{p^2_{21}} \right] s = 0, \end{aligned}$$

where

$$\begin{aligned} \pi &= 1/p_{12} (p^3_{12} p_{21} - 4\Delta_1), \quad \bar{\pi} = 1/p_{12} (p^3_{12} p_{21} - 4\bar{\Delta}_1), \\ x &= 1/p_{21} (p_{12} p^3_{21} - 4\Delta_2), \quad \bar{x} = 1/p_{21} (p_{12} p^3_{21} - 4\bar{\Delta}_2), \\ \Delta_1 &= p^2_{12} q_{11} - p_{12} q'_{12} + 3q^2_{12}, \quad \bar{\Delta}_2 = p^2_{21} q_{22} - p_{21} q'_{21} + 3q^2_{21}. \end{aligned}$$

The tetrahedron $P_u P_v P_r P_s$ is related to the two complexes L, L_1 , in an interesting way. The two edges l_{uv}, l_{rs} have just been shown to belong to both L and L_1 . By (20) the general point $(\alpha, 0, \beta, 0)$ on l_{ur} corresponds, in the null-system of L , to the plane $\beta x_1 - \alpha x_3 = 0$. But this is a plane on the line l_{vs} . Similarly the general point on l_{vs} can be shown to correspond to a

* *Proj. Dif. Geom.*, p. 216.

plane on l_{ur} . These two lines are therefore polar reciprocals with respect to L . By making use of (12) we find that l_{ur} and l_{vs} belong to L_1 . Similarly we find that the remaining two edges l_{us} , l_{vr} , belong to L and are polar reciprocals with respect to L_1 . Of the six edges then, two opposite edges belong to both L and L_1 , two belong to L_1 and are polar reciprocals with respect to L , and two belong to L and are polar reciprocals with respect to L_1 . It may be noted further that the two vertices P_v , P_r , lie upon C_F'' (for $t = 0, \infty$) and the other two, P_u , P_s , lie upon C_F'' .

This tetrahedron $P_u P_v P_r P_s$ is of sufficient importance to warrant us in making use of it as a new tetrahedron of reference. We will obtain the equations of transformation and apply them in a number of instances.

From (26) and (27) we find

$$(31) \quad \begin{aligned} 2Py &= p_{12}r - p_{21}u, & 2Pz &= p_{12}s - p_{21}v, \\ 2P\rho &= 2(q_{21}u - q_{12}r), & 2P\sigma &= 2(q_{21}v - q_{12}s). \end{aligned}$$

If $\underline{x}_1, \dots, \underline{x}_4$ are the coördinates of any point referred to $P_u P_v P_r P_s$ and x_1, \dots, x_4 , the coördinates of the same point referred to $P_v P_z P_\rho P_\sigma$, so that

$$\underline{x}_1 u + \underline{x}_2 v + \underline{x}_3 r + \underline{x}_4 s = x_1 y + x_2 z + x_3 \rho + x_4 \sigma,$$

then from (31) we have, except for a proportionality factor,

$$(32) \quad \begin{aligned} \underline{x}_1 &= -p_{21}x_1 + 2q_{21}x_3, & \underline{x}_2 &= -p_{21}x_2 + 2q_{21}x_4, \\ \underline{x}_3 &= p_{12}x_1 - 2q_{12}x_3, & \underline{x}_4 &= p_{12}x_2 - 2q_{12}x_4, \end{aligned}$$

or, solving for x_1, \dots, x_4 and again omitting a factor $2P$,

$$(32') \quad \begin{aligned} x_1 &= 2(q_{12}\underline{x}_1 + q_{21}\underline{x}_3), & x_2 &= 2(q_{12}\underline{x}_2 + q_{21}\underline{x}_4), \\ x_3 &= p_{12}\underline{x}_1 + p_{21}\underline{x}_3, & x_4 &= p_{12}\underline{x}_2 + p_{21}\underline{x}_4. \end{aligned}$$

(32) and (32') are projective transformations, inverse to each other. The corresponding transformation in line coördinates is

$$(33) \quad \begin{aligned} \omega_{12} &= 4[q_{12}^2 \omega_{12} + q_{12}q_{21}(\omega_{14} - \omega_{23}) + q_{21}^2 \omega_{34}], \\ \omega_{13} &= -2P\omega_{13}, \\ \omega_{14} &= 2(p_{12}q_{12}\omega_{12} + p_{21}q_{12}\omega_{14} - p_{12}q_{21}\omega_{23} + p_{21}q_{21}\omega_{34}), \\ \omega_{23} &= 2(-p_{12}p_{12}\omega_{12} - p_{12}q_{21}\omega_{14} + p_{21}q_{12}\omega_{23} - p_{21}q_{21}\omega_{34}), \\ \omega_{42} &= -2P\omega_{42}, \\ \omega_{34} &= p_{12}^2 \omega_{12} + p_{12}p_{21}(\omega_{14} - \omega_{23}) + p_{21}^2 \omega_{34}. \end{aligned}$$

Applying (32), (32'), (33) to a number of previous results we find, for the new parametric equations of C_F' ;

$$(34) \quad \xi_1 = -p_{21}^2 t, \quad \xi_2 = -p_{21}^2, \quad \xi_3 = p_{12}^2 t^3, \quad \xi_4 = p_{12}^2 t^2;$$

for C_F'' ,

$$(35) \quad \eta_1 = p^3_{21}, \quad \eta_2 = 3p_{12}p^2_{21}t, \quad \eta_3 = 3p^2_{12}p_{21}t^2, \quad \eta_4 = p^3_{12}t^3;$$

for the new equation of Q_1 ,

$$(36) \quad p^3_{12}x^2_1 + p^2_{12}p_{21}x_1x_3 + p_{12}p^2_{21}x_2x_4 + p^3_{21}x^2_4 = 0;$$

and for L_1 ,

$$3\omega_{14} - \omega_{23} = 0.$$

The equations of the osculating quadric Q and of the osculating linear complex are unchanged.

IV. THE COMPLEX CONES AND THE CONE CUBICS.

Let us think of a point P_1 ; $(\alpha, \beta, 0, 0)$ as moving along l_{yz} from P_y to P_z while at the same time a corresponding point P_2 ; $(0, \alpha, \beta, 0)$ moves along $l_{z\rho}$ from P_z to P_ρ . The equations of the pair of planes corresponding to these two points in the null-system of L are, by (20), respectively

$$(37) \quad \alpha p_{12}x_3 - \beta p_{21}x_4 = 0, \quad \beta p_{12}x_1 + \alpha p_{21}x_4 = 0.$$

As P_1, P_2 , move, these planes rotate about the respective lines $l_{yz}, l'_{z\rho}$, corresponding pairs of planes in the two pencils intersecting in the lines of a quadric cone whose vertex is P_z . The equation of this cone, referred to the tetrahedron $P_yP_zP_\rho P_\sigma$ is found by eliminating α, β from equations (37). It is

$$(38) \quad p^2_{12}x_1x_3 + p^2_{21}x^2_4 = 0.$$

At the same time that P_2 moves along $l_{z\rho}$ from P_z to P_ρ , let the point P_3 ; $(0, 0, \alpha, \beta)$ move along $l_{\rho\sigma}$ from P_ρ to P_σ . The plane corresponding to P_3 in the null-system of L has for its equation

$$(39) \quad \alpha p_{12}x_1 - \beta p_{21}x_2 = 0.$$

As P_3 moves, this plane rotates about $l_{\rho\sigma}$, in each of its positions corresponding to a plane of the pencil whose axis is $l_{z\rho}$. There is thus generated a second quadric cone whose vertex is P_ρ and whose equation is found by eliminating α, β from (39) and (38). It is

$$(40) \quad p^2_{12}x^2_1 + p^2_{21}x_2x_4 = 0.$$

These two cones have the line $l_{z\rho}$ in common but they are not tangent along this line, the tangent planes along this common element being respectively the planes $P_zP_\rho P_\sigma$ and $P_yP_zP_\rho$. Their complete intersection proves to

be the line l_{zp} and a space cubic. For the parametric equations of this cubic let us choose arbitrarily $x_1 = p^2_{21}\lambda\mu^2$, $x_2 = p^2_{21}\mu^3$. Substituting these values in (38) and (40) we find for the other two coördinates $x_3 = -p^2_{12}\lambda^3$, $x_4 = -p^2_{12}\lambda^2\mu$. If we resort to a non-homogeneous parametric form, writing $\lambda/\mu = -t$, then our cubic is expressible by the equations

$$(34) \quad x_1 = -p^2_{21}t, \quad x_2 = -p^2_{21}, \quad x_3 = p^2_{12}t^3, \quad x_4 = p^2_{12}t^2.$$

If we use the three lines l_{zy} , $l_{y\sigma}$, $l_{\sigma p}$, rather than the set l_{yz} , l_{zp} , $l_{p\sigma}$, we obtain two cones again intersecting in a common element $l_{y\sigma}$ and in a space cubic. The equations of the cones are

$$(41) \quad p^2_{12}x^2_3 + p^2_{21}x_2x_4 = 0, \quad p^2_{12}x_1x_3 + p^2_{21}x^2_2 = 0,$$

and the parametric equations of the cubic may be written

$$(42) \quad x_1 = p^2_{12}t^3, \quad x_2 = p^2_{12}t^2, \quad x_3 = -p^2_{21}t, \quad x_4 = -p^2_{21}.$$

Each of the four cones already considered has been defined by means of the osculating linear complex L and a pair of coplanar edges of $P_yP_zP_\rho P_\sigma$ belonging to L . Only four of the six edges of $P_zP_yP_\rho P_\sigma$ can be used in this way since l_{yp} , $l_{z\sigma}$ do not belong to L . The four remaining edges can be paired in six different ways but in only four of the six ways do the paired lines intersect. The four cones (38), (40), (41₁), (41₂) are therefore all that can be defined in this manner. We shall speak of these cones as the *complex cones associated with g*.

Each of the cubics (34), (42) is determined by a pair of the complex cones. These four cones can be paired in six ways but the intersection of the pair (38), (41₂) degenerates into two conics. This is also true of the pair (40), (41₁). The four complex cones therefore determine but four space cubics. We shall speak of these as the *cone cubics associated with g*. We have written parametric equations for two of the four. We find by inspection for the cubic determined by cones (38) and (41₁), the equations

$$(43) \quad x_1 = p^2_{12}t^3, \quad x_2 = -p^2_{21}, \quad x_3 = -p^2_{21}t, \quad x_4 = p^2_{12}t^2,$$

and for that determined by (40) and (41₂), the equations

$$(44) \quad x_1 = -p^2_{21}t, \quad x_2 = p^2_{12}t^2, \quad x_3 = p^2_{12}t^3, \quad x_4 = -p^2_{21}.$$

of these four cubics, two pass through each vertex of the tetrahedron $P_yP_zP_\rho P_\sigma$. Two of these cubics lie upon the osculating quadric Q , as may be verified.

We conclude this section by indicating the relation between the primary and secondary flecnodal cubics and the cone cubics. In the first place the

parametric equations of the primary flecnodal cubic referred to the tetrahedron $P_u P_v P_r P_s$ are precisely those of the first cone cubic when referred to system $P_y P_z P_\rho P_\sigma$. This leads us to remark that the two sets of parametric equations (C_F'') and (34) may be thought of either as defining a single curve referred to two distinct reference systems, or as defining two distinct curves referred to the same reference system. Adopting this second alternative and remembering the projective nature of transformation (32), we find that *the first of the four cone cubics is projectively equivalent to the primary flecnodal cubic.*

In setting up transformation (32) the correspondence between the vertices of the two reference tetrahedrons was chosen in an arbitrary manner, the order (y, z, ρ, σ) corresponding to the order (u, v, r, s) . By this choice vertices of one tetrahedron were made to correspond to vertices of the other lying on the same flecnodal tangent. We may rearrange the vertices of the second tetrahedron in three ways and still keep corresponding vertices on the same flecnodal tangent, thus, (r, s, u, v) , (r, v, u, s) , (u, s, r, v) . From these arrangements three additional transformations of type (32) are determined, and these, when applied to (C_F'') transform it into the remaining three cone cubics in the order (42), (43), (44). *The primary flecnodal cubic and the four cone cubics associated with each ruling g of S are therefore all projectively equivalent.*

Since C_F' is determined as the intersection of Q and Q_1 the cone cubic (34) must be determined as the intersection of Q and the quadric (36) into which Q_1 is transformed by (32). This quadric is in fact that one of the pencil of quadrics

$$(45) \quad \kappa(p_{12}^2 x_1^2 + p_{21}^2 x_2 x_4) + \lambda(p_{12}^2 x_1 x_3 + p_{21}^2 x_2^2) = 0,$$

determined by the two complex cones (38) and (40), given by $\kappa : \lambda = p_{12} : p_{21}$. Each quadric of the pencil (45) has in common with Q the first cone cubic (34) and a straight line which intersects the flecnodal tangents in the points $(\lambda, 0, -\kappa, 0)$ $(0, \lambda, 0, -\kappa)$. The cone cubic (42) is the intersection of Q and that one of the pencil of quadrics

$$(46) \quad \kappa(p_{12}^2 x_3^2 + p_{21}^2 x_2 x_4) + \lambda(p_{12}^2 x_1 x_3 + p_{21}^2 x_2^2) = 0,$$

determined by the complex cones (41₁), (41₂), for which $\kappa : \lambda = p_{12} : p_{21}$. Each quadric of the pencil (46) has in common with Q the second cone cubic (40) and a straight line which intersects the flecnodal tangents in the points $(\kappa, 0, -\lambda, 0)$, $(0, \kappa, 0, -\lambda)$.

If we identify the parameters in the three equations (C_F''), (34) and (42), we shall find that the point correspondence thus established between

these curves is precisely that determined by the lines of that regulus of Q to which the flecnodal tangents belong. Those points of the three curves correspond which lie upon the same line of this regulus.

To the cone cubics there correspond by means of their osculating planes and the null-system of L four cubics which bear to the cone cubics the same relation as does the secondary flecnodal cubic to the primary flecnodal cubic. The parametric equations of that obtained from (34) are indeed given by (35) and those for the remaining three cone cubics may be obtained from (35) by interchanging the values of x_1 and x_3 , x_2 and x_4 , both pairs or pair at a time. The four cubics thus obtained are all projectively equivalent to C_F , the projective transformations being those which carry C_F into the four cone cubics.

V. CONCLUSION. EXTENSION OVER S .

The developments in sections III and IV have been concerned for the most part with the properties of lines, curves and surfaces associated with the primary and secondary flecnodal cubics. Symmetrical developments might have been obtained for the complex cubics. We might have obtained, for instance, the directrices of the congruence Γ_2 , the two lines which it has in common with Q and properties of the tetrahedron formed by these two lines with the lines l_{yp} , l_{zs} . Using this tetrahedron as a new system of reference we might have arrived at new equations for the curves C_C and C_C' , interpreting these again as equations of new curves referred to the original reference system. The obvious method of securing such results is the application of the projective transformation (6) to the equations already obtained. There is no necessity for pursuing this matter further at this time.

It is essential to note that with one exception, equations (29) and (30), we have concerned ourselves only with the independent variable t . Geometrically speaking we have remained on our ruled surface S in the neighborhood of the ruling g . If now we think of x as also varying we thereby move from line to line of S carrying with us those lines, curves, surfaces and congruences which have been associated with g . There will thus be generated by each line a ruled surface, by each space curve a curved surface, by each ruled surface a line congruence, and by each congruence a line complex.

We have noted that each of the cone cubics is projectively equivalent to the primary flecnodal cubic. It does not follow however that the surfaces generated by the cone cubics will be projectively equivalent to that generated by the primary flecnodal cubic. For the coefficients of the four transformations of which (32) is one, are constants only as concerns t , so that in general

a single projective transformation does not exist which will carry the surface of primary flecnodal cubics into a surface of cone cubics. It is true however that to each flecnodal cubic on the former surface there corresponds a projectively equivalent cone cubic on the latter. In a similar manner the complexes generated by Γ_1 and Γ_2 are not projectively equivalent although they consist of two one-parameter families of congruences which are so related that to each congruence of one there corresponds a projectively equivalent congruence in the other.

The projective differential geometry as developed by Wilczynski, Green and others provides most of the theory necessary for adequate investigation of the properties of the surfaces, congruences, and complexes obtained by extending over the ruled surface S the configurations defined in the preceding sections. Their further investigation would seem well worth the effort required.

UNIVERSITY OF WASHINGTON,
November 6, 1923.

Complete Characterization of Dynamical Trajectories in N -Space.

BY LYMAN M. KELLS.

The object of the present paper is to work out for n dimensions a theory analogous to that worked out by Professor Kasner, of Columbia University, for two dimensions* and three dimensions.†

The paper considers in n -space an arbitrary field of force in which the motion of a freely moving particle is defined by the following equations:

$$(1) \quad \ddot{x}_1 = \psi_1(x_1 x_2 \cdots x_n), \quad \ddot{x}_2 = \psi_2(x_1, x_2 \cdots x_n), \cdots \\ \ddot{x}_n = \psi_n(x_1 x_2 \cdots x_n)$$

where the dots denote derivatives with respect to the time and the ψ functions are arbitrary functions of the n variables $x_1 \cdots x_n$ possessing derivatives of the first and second order in the region of space considered.

From each point a particle may be sent out in any one of the ∞^{n-1} directions with any one of the ∞^1 possible speeds. Consequently passing through each point there will be ∞^n curves, ∞^1 in each of the ∞^{n-1} directions. The entire system of curves is then $(2n-1)$ -fold infinite.‡ We wish to find a set of purely geometric properties which shall completely characterize this system of curves.

DIFFERENTIAL EQUATIONS OF N -SPACE TRAJECTORIES.

1. To get purely geometric properties it is necessary to eliminate the time. Combination of equations (1) with equations

$$(2) \quad \dot{x}_i = \frac{dx_i}{dt}, \quad x'_i = \frac{dx_i}{dx_1} = \frac{\dot{x}_i}{\dot{x}_1}, \quad x''_i = \frac{d^2x_i}{dx_1^2} = \frac{\dot{x}_1 \ddot{x}_i - \dot{x}_i \ddot{x}_1}{\dot{x}_1^3} \quad (i = 1 \cdots n),$$

gives equations

$$(3) \quad \dot{x}_1^2 = \frac{\psi_i - x'_i \psi_1}{x''_i} = \frac{\psi_r - x'_r \psi_1}{x''_r} \quad (i = 2 \cdots n, r = 2 \cdots n).$$

* "Trajectories of Dynamics," E. Kasner, *Transactions*, Vol. 7 (1906), pp. 401-424. This paper will be referred to as "Trajectories."

† "Dynamical Trajectories: The Motion of a Particle in an Arbitrary Field of Force," E. Kasner, *Transactions*, Vol. 8 (1907), pp. 135-158.

‡ This system of curves we call the System of dynamical trajectories in n -space.

Taking the derivative of (3), we obtain

$$(4) \quad x_r''' = \left\{ \left(\sum_{i=1}^n x'_i \psi_{rx_i} - x'_r \sum_{i=1}^n x'_i \psi_{ix_i} \right) x_r'' - 3\psi_1 x_r''^2 \right\} \\ \div (\psi_r - x'_r \psi_1) \quad (r = 2 \cdots n).$$

where the variable subscripts denote partial derivatives. Equations (3) and the first one of (4) ($r=2$) for assigned values of $x_1 \cdots x_n, x'_2 \cdots x'_n, x_2''$ determine direction and initial velocity, i. e. a trajectory. Hence the defining equations of our system may be taken as

$$(5) \quad x_2''' = Px_2'' + Qx_2''^2, \quad x_i'' = K_i x_2'' \quad (i = 3 \cdots n),$$

where

$$(5') \quad K_i = \frac{\psi_i - x'_i \psi_1}{\psi_2 - x'_2 \psi_1} \quad (i = 3 \cdots n), \quad P = \frac{\sum_{i=1}^n x'_i \psi_{2x_i} - x'_2 \sum_{i=1}^n x'_i \psi_{ix_i}}{\psi_2 - x'_2 \psi_1}, \\ Q = \frac{-3\psi_1}{\psi_2 - x'_2 \psi_1}$$

2. By introducing the equations

$$(6) \quad \omega_i = \psi_{i+1} \div \psi_1, \quad \phi = \log \psi_1$$

we may write the equations of the lines of force as

$$(7) \quad dx_1 : dx_2 : \cdots : dx_n = \psi_1 : \psi_2 : \cdots : \psi_n = 1 : \omega_1 : \cdots : \omega_{n-1},$$

and the coefficients P_r and Q_r of x''_r and x''^2_r respectively in (4) as

$$(8) \quad P_r = \sum_{i=1}^n x'_i \phi_{x_i} - \frac{\sum_{i=1}^n x'_i \omega_{(r-1)x_i}}{x'_r - \omega_{r-1}}, \quad Q_r = \frac{3}{x'_r - \omega_{r-1}}, \quad K_r = \frac{x'_r - \omega_{r-1}}{x'_2 - \omega_1}, \\ (r = 2 \cdots n).$$

OSCULATING PLANES.

3. The first property of n -dimensional dynamical trajectories comes from a consideration of the osculating planes of the curves passing through a given point. If the point is taken as origin and $x_1 \cdots x_n$ as current co-ordinates the osculating plane* is given by the equations represented by

$$(9) \quad \begin{vmatrix} X_1 & X_2 & \cdots & X_n \\ 1 & x'_2 & \cdots & x'_n \\ 0 & x''_2 & \cdots & x''_n \end{vmatrix} (=0) \quad \dagger$$

* *Repertorium der höheren Mathematik*, Vol. 2, E. Pascal, p. 599.

† The symbol (9) will be used as above throughout the paper.

where the symbol (9) stands for the equations arising from equating each three-columned minor of the matrix to zero. Using equations (3) we find that this is identically satisfied for all values of $x'_2 \cdots x'_n$ by $X_1 : X_2 : \cdots : X_n = 1 : \omega_1 : \omega_2 : \cdots : \omega_{n-1}$. Hence,

Property I. The osculating planes, at a given point, of all dynamical trajectories in n -space through that point, form a hyper-pencil of planes.

The axis of the hyperpencil passes through the origin in the direction of the force acting at the point.

4. To find the equations defining all $(2n-1)$ -fold infinite systems of curves possessing property I, we begin by letting $1 : \omega_1 : \cdots : \omega_{n-1}$, where the ω_i are arbitrary functions of $x_1 \cdots x_n$, define the direction of the axis of the hyperpencil. Then the equations represented by

$$\begin{vmatrix} 1 & \omega_1 & \omega_2 & \cdots & \omega_{n-1} \\ 1 & x'_2 & x'_3 & \cdots & x'_n \\ 0 & x''_2 & x''_3 & \cdots & x''_n \end{vmatrix} (=0)$$

must hold for all values of $x'_2 \cdots x'_n$. This gives the relations

$$x''_i = \frac{x'_i - \omega_{i-1}}{x'_2 - \omega_1} x''_2, \quad (i = 2 \cdots n).$$

Conversely, when these equations are satisfied the system of curves has property I. This proves the statement.

The most general $(2n-1)$ -fold infinite system of curves in n -space possessing property I is represented by equations having the form

$$(10) \quad x_2''' = f(x_1 \cdots x_n x'_2 \cdots x'_n x''_2),$$

$$x''_i = \frac{x'_i - \omega_{i-1}}{x'_2 - \omega_1} x''_2 \quad (i = 3 \cdots n).$$

5.* From the general expressions for the osculating lineoid and the second radius of curvature, it is not difficult to deduce the following theorems.†

All curves belonging to a $(2n-1)$ -fold infinite system which is represented by equations of the form (10), and passing through a given point in a given direction, have a common osculating plane at the given point.

* The theorems here stated are not necessary for the characterization.

† Some results related to these are stated by Professor Kasner in "The Princeton Colloquium Lectures on Mathematics." Published by American Mathematical Society, New York, 1913.

The ∞^1 dynamical trajectories in n -space passing through a given point in a given direction, have at that point a common osculating lineoid (3-flat) and a common second radius of curvature.

The dynamical trajectories in n -space which pass through a given point and have there the same second radius of curvature c , touch at the given point the quartic hypercone

$$(11) \quad \left[\begin{vmatrix} X_1 & X_2 & \cdots & X_n \\ 1 & \omega_1 & \cdots & \omega_{n-1} \end{vmatrix}^2 - c^2 \begin{vmatrix} X_1 & X_2 & X_3 & \cdots & X_n \\ 1 & \omega_1 & \omega_2 & & \omega_{n-1} \\ 0 & \sum_{i=1}^n X_i \omega_{1i} & \sum_{i=1}^n X_i \omega_{2i} & \cdots & \sum_{i=1}^n X_i \omega_{(n-1)i} \end{vmatrix}^2 \right] = 0,$$

where the matrix with the number 2 at the upper right hand corner indicates the sum of the squares of its minors.

Each hypercone (11) contains the line $(s, s\omega_1, \cdots s\omega_{n-1})$, (s variable) as a cuspidal edge.

Of the ∞^n dynamical trajectories in n -space through a given point, there are ∞^{n-1} which have hyperosculating planes. The lineal elements of these determine the quadric hypercone

$$(12) \quad \left\| \begin{vmatrix} X_1 & X_2 & \cdots & X_n \\ 1 & \omega_1 & \cdots & \omega_{n-1} \end{vmatrix} \right\|^2 = 0.$$

OSCULATING HYPERSPHERES.

6. The second property to be derived relates to the centers of the hyperspheres which osculate the curves of the system to the third order. If $X_1, X_2, \cdots X_n$ be the center of the hypersphere and if we take the origin at the point to be considered, the equation of the hypersphere may be written in the form

$$2X_1x_1 + 2X_2x_2 + \cdots + 2X_nx_n = \sum_{i=1}^n x_i^2.$$

If this is to osculate a trajectory to the third order, the 1st, 2d and 3d derivatives of the variables of the hypersphere must be equal to the 1st, 2d and 3d derivatives respectively of the trajectory. This gives the conditions

$$(13) \quad \begin{aligned} X_1 + x'_2X_2 + x'_3X_3 + \cdots + x'_nX_n &= 0 \\ x_2''(X_2 + K_3X_3 + \cdots + K_nX_n) &= 1 + \sum_{i=2}^n x_i'^2 \\ x_2'''X_2 + x_3'''X_3 + \cdots + x_n'''X_n &= 3\sum_{i=2}^n x_i'x_i''. \end{aligned}$$

Elimination from (13) of X_2'' and X_i''' by means of equations (5) gives the following equations free from X_2'' :

$$(14) \quad \begin{aligned} X_1 + x'_2 X_2 + x'_3 X_3 + \cdots + x'_n X_n &= 0 \\ P X_2 + P_3 k_3 X_3 + \cdots + P_n k_n X_n + Q \sum_1^n x_i'^2 - 3(x'_2 + \sum_3^n k_i x_i') &= 0 \end{aligned}$$

As the coefficients are constants and the equations are linear in the X_i , we have

Property II. The centers of the third order osculating hyperspheres at a point of the ∞^1 dynamical trajectories in n -space passing through that point in a given direction, lie in an $(n-2)$ -flat.

7. *Converse of Property II.* We now find the form of the defining equations of all $(2n-1)$ -fold infinite systems of curves possessing properties I and II. As the $(n-2)$ -flat in which the 3rd order osculating hypersphere centers are to lie must be contained in the $(n-1)$ -flat normal to the element, its equations are of the form

$$(15) \quad \sum_1^n x_i' X_i = 0, \quad \sum_1^n A_i X_i + A_{n+1} = 0$$

where the A_i involve $x_1 \cdots x_n, x'_2 \cdots x'_n$ in any way. If we now substitute equations (10) in equations (13), solve the results for X_2, X_3, X_4 and substitute in (15), the result must be identically zero for all values of $X_1, X_5, X_6 \cdots X_n$. Equating the absolute term to zero we find $f = X_2''' = G X_2'' + H X_2''^2$ where G and H are arbitrary functions of the x 's and x'' 's. Moreover it is easy to show that if $x_2''' = G x_2'' + H x_2''^2$, property II holds. Hence

The most general $(2n-1)$ -fold infinite system of n -space curves possessing properties I and II are defined by equations of the form

$$(16) \quad x_i'' = k_i x_2'' = \frac{x'_i - \omega_{i-1}}{x'_2 - \omega_1} x_2'' \quad (i = 3 \cdots n), \quad x_2''' = G x_2'' + H x_2''^2.$$

8. It is not difficult to derive the following incidental results:

For any given point the locus of the points of intersection of the osculating planes of the trajectories through the point, with the corresponding $(n-2)$ -flats of hypersphere centers, is a curved space represented by an n -th degree equation having no terms of degree less than $n-1$. The locus contains each point of the line $X_1/\psi_1 = X_2/\psi_2 = \cdots = X_n/\psi_n$ as the vertex of a

hypercone whose elements have contact of the $(n-1)$ -st order with the considered locus.*

THE $(n-1)$ -SET Ω AND THE n -IC Γ .

9. The assemblage of the ∞^{n-1} $(n-2)$ -flats associated according to property II to the ∞^{n-1} lineal elements at a point, called the $(n-1)$ -set Ω is defined by equations (14) which may be written in the form

$$(17) \quad \sum_{i=1}^n x_i' X_i = 0, \quad \sum_{i=1}^n x_i' Y_i = 0,$$

where

$$(17') \quad Y_k = \sum_{i=1}^n X_i \psi_{ix_k} - X_k \psi_{1x_1} - 3\psi_k \quad (k=1 \dots n)$$

Elimination of x'_k from (17) gives

$$(18) \quad \sum_{r=1}^n x'_r [X_r Y_k - X_k Y_r] = 0$$

Plainly the curve, called the n -ic Γ , satisfying the equations $X_r Y_k - X_k Y_r = 0$ or

$$(19) \quad X_1/Y_1 = X_2/Y_2 = \dots = X_n/Y_n \quad (= \text{a variable } T)$$

will be contained in every $(n-1)$ -flat represented by equation (18). We can solve the equations $X_k = T Y_k$ ($k=1 \dots n$) simultaneously and thus obtain the parametric equations of the n -ic Γ in the form

$$(20) \quad X_k = \sum_{i=0}^n a_i^{(k)} T^i \quad (k=1 \dots n)$$

Since these equations are of the n -th degree in T , the n -ic Γ is a twisted curve in n -space of the n -th order.

Since from equations (19) we can derive

$$(21) \quad \sum_{i=1}^n x'_i X_i = T \sum_{i=1}^n x'_i Y_i$$

it appears that any point except the origin which satisfies the first of equations (17) and equations (19) will also satisfy the second equation of (17), i. e. every $(n-2)$ -flat (17) meets the n -ic Γ in $(n-1)$ -points. Moreover, through any $n-1$ points of the n -ic Γ passes an $(n-2)$ -flat of the set Ω , since if the x'_i are so chosen that the $(n-1)$ points lie in the first $(n-1)$ -

* It is interesting to notice that this theorem has a non-trivial meaning first in 4-space.

flat of (17), they will also lie in the second on account of equation (21). Hence

The $(n-1)$ -set Ω is composed of the secant $(n-2)$ -flats of a twisted n -space curve of the n -th order.

If we take as the equations of the n -ic Γ

$$(22) \quad f_i = X_i Y_{i+1} - X_{i+1} Y_i = 0 \quad [i = 1 \cdots (n-1)]$$

the equations of the tangent at the origin are given by $\sum_{i=1}^n X_i \frac{\partial f_i}{\partial X_i^0} = 0$, where the zero superscripts mean that the X_i are to be replaced by zero in the partial derivatives. This gives as the equation of the tangent to (22)

$$(23) \quad X_1 : X_2 : \cdots : X_n = \psi_1 : \psi_2 : \cdots : \psi_n = 1 : \omega_1 : \omega_2 : \cdots : \omega_{n-1}.$$

We have then

Property III. The ∞^{n-1} $(n-2)$ -flats, each of which correspond according to property II to one of the ∞^{n-1} lineal elements at a point, make up the secant $(n-2)$ -flats of an n -th order n -space curve which passes through the considered point in the direction of the force acting at the point.

10. Converse of III.

In order that the system of curves possess properties I and II its defining equations must have the form,

$$(24) \quad x_2''' = Gx_2'' + Hx_2''^2, \quad x_i'' = k_i x_2'' \quad (i = 3 \cdots n).$$

The equations defining the $(n-2)$ -flat of centers of third order osculating hyperspheres in the case of the system (16) are

$$(25) \quad \sum_1^n x'_i X_i = 0, \quad \sum_2^n M_i X_i + 1 = 0$$

where

$$(25') \quad M_i = \frac{k_i G + k'_i}{H \sum_1^n x'_i{}^2 + 3 \sum_2^n x'_i k_i} \quad *$$

The homogeneous parametric equations of the twisted n -ic curve associated to a given point have the form

$$(26) \quad \begin{aligned} X_i &= a_n^{(i)} \lambda^n + a_0^{(i)} \lambda^{n-1} + a_1^{(i)} \lambda^{n-2} + \cdots + a_{n-2}^{(i)} \lambda + a_{n-1}^{(i)} \\ T &= a_n^{(n+1)} \lambda^n + a_0^{(n+1)} \lambda^{n-1} + a_1^{(i)} \lambda^{n-2} + \cdots + a_{n-2}^{(n+1)} \lambda + a_{n-1}^{(n+1)} \end{aligned} \quad (i = 1 \cdots n)$$

* Here $K_2 \equiv 1$, $K'_2 = 0$.

Where the $a_i^{(k)}$ are arbitrary functions of $x_1 \cdots x_n$. We may take the considered point as the origin and make the value $\lambda = \infty$ correspond to it by placing

$$(27) \quad a_n^{(n+1)} = 1, \quad a_n^{(i)} = 0 \quad (i = 1 \cdots n).$$

Also since the curve is to pass through the origin in the direction $1 : \omega_1 : \omega_2 : \cdots : \omega_{n-1}$ we must have,

$$(28) \quad a_0^{(1)} : a_0^{(2)} : \cdots : a_0^{(n)} = 1 : \omega_1 : \omega_2 : \cdots : \omega_{n-1}.$$

Substitution of the values (26) in the equations (25) taking account of (27) gives

$$(29) \quad L_0 \lambda^{n-1} + L_1 \lambda^{n-2} + \cdots + L_{n-2} \lambda + L_{n-1} = 0 \\ \lambda^n + A_0 \lambda^{n-1} + A_1 \lambda^{n-2} + \cdots + A_{n-2} \lambda + A_{n-1} = 0$$

where

$$(29') \quad L_i = \sum_{t=1}^n a_i^{(t)} x'_t, \quad A_i = \sum_{t=2}^n M_t a_i^{(t)} + a_i^{(n+1)}, \quad [i = 0, 1, 2, \cdots (n-1)]$$

According to property III equations (25) must have $n-1$ common solutions with (26) and this imposes the following $n-1$ conditions:

$$(30) \quad L_0 M_2 (L_0 a_i^{(2)}) + L_0 M_3 (L_0 a_i^{(3)}) + \cdots + L_0 M_n (L_0 a_i^{(n)}) \\ = L_0 L_{i+1} - L_1 L_i - L_0 (L_0 a_i^{(n+1)}), \quad L_n = 0, \quad [i = 1 \cdots (n-1)],$$

where the parentheses represent determinants in terms of their principal diagonals, thus $(L_0 a_1^{(2)}) = L_0 a_1^{(2)} - L_1 a_0^{(2)}$.

The solution of the equations (30) for M_i is found first in determinant form and then simplified to

$$(31) \quad M_i = \frac{\sum_{t=1}^n l_t^{(i)} x'_t - x'_i \sum_{t=2}^n l_t^{(i)} x'_t}{3(1 + x'_2 \omega_1 + \cdots + x'_n \omega_{n-1})} \quad (i = 2 \cdots n)$$

where $l_t^{(i)}$ represents arbitrary functions of $x_1 \cdots x_n$.

It is easy to show that conversely if the functions M_i have the form (31) they define a system of curves possessing properties I, II and III.

In summary then

The most general $(2n-1)$ -fold infinite system of curves possessing properties I, II and III are defined by equations which have the form

$$(32) \quad x_2''' = Gx_2'' + Hx_2''^2, \quad x_i'' = k_i x_2'' \quad (i = 3 \cdots n),$$

where

$$(32') \quad k_i = \frac{x'_i - \omega_{i-1}}{x'_2 - \omega_1} \quad (i = 3 \cdots n)$$

and the functions G and H satisfy the relations

$$\begin{aligned}
 (33) \quad G : k_3 G + k'_3 : \dots : k_n G + k'_n : H \sum_1^n x'_t{}^2 - 3(x'_2 + \sum_3^n k_t x'_t) \\
 = \sum_1^n l_t^{(2)} x'_t - x'_2 \sum_2^n l_t^{(1)} x'_t : \sum_1^n l_t^{(3)} x'_t - x'_3 \sum_2^n l_t^{(1)} x'_t : \dots \\
 : \sum_1^n l_t^{(n)} x'_t - x'_n \sum_2^n l_t^{(1)} x'_t : 3(1 + x'_2 \omega_1 + \dots + x'_n \omega_{n-1})
 \end{aligned}$$

11. We now proceed to find the relations which must exist among the $(n-1)(n+2)$ arbitrary functions involved in (33) in order that the system represented be a dynamical system.

In the second equation (17) make the change of variable indicated in article 2 and then compare coefficients with those of the corresponding equation for the system defined by (32) and (33). This gives the necessary relations

$$\begin{aligned}
 (34) \quad -l_i^{(1)} &= \phi_{x_i} \quad (i = 2 \dots n), \\
 l_i^{(j+1)} &= -\omega_j x_i - \omega_j \phi_{x_i} \quad [j = 1 \dots (n-1), i = 1 \dots n, i \neq j] \\
 l_i^{(4)} &= -\omega_{(i-1)} x_i - \omega_{i-1} \phi_{x_i} + \phi_{x_1} \quad (i = 2 \dots n).
 \end{aligned}$$

The explicit conditions are obtained by eliminating ϕ from equations (34). This gives

$$\begin{aligned}
 (35) \quad \frac{l_i^{(j)} + \omega_{(j-1)} x_i}{\omega_{j-1}} &= \frac{l_i^{(t)} + \omega_{(t-1)} x_i}{\omega_{t-1}} = l_i^{(4)} \\
 (i = 2 \dots n, j = 2 \dots n, t = 2 \dots n, i \neq j, i \neq t) \\
 l_2^{(2)} + \omega_{1x_2} - \omega_1 l_2^{(1)} &= l_3^{(3)} + \omega_{2x_3} - \omega_2 l_3^{(1)} = \dots \\
 &= l_n^{(n)} + \omega_{(n-1)x} - \omega_{n-1} l_n^{(1)} \\
 &= -\frac{l_1^{(2)} + \omega_{1x_1}}{\omega_1} = -\frac{l_1^{(3)} + \omega_{2x_2}}{\omega_2} = \dots = -\frac{l_1^{(n)} + \omega_{(n-1)x_1}}{\omega_{n-1}}
 \end{aligned}$$

The conditions of integrability of the equations

$$(36) \quad \phi_{x_i} = \frac{l_1^{(i)} + \omega_{(i-1)} x_1}{-\omega_{i-1}} \quad (i = 2 \dots n), \quad -l_i^{(1)} = \phi_{x_i} \quad (i = 2 \dots n),$$

are

$$(37) \quad l_{ix_1}^{(1)} - \left[\frac{l_1^{(i)} + \omega_{(i-1)} x_1}{\omega_{i-1}} \right]_{x_1} = 0, \quad (i = 2 \dots n)$$

and

$$(38) \quad l_{ix_j}^{(1)} = l_{jx_i}^{(1)} \quad (i = 2 \dots n, j = 2 \dots n).$$

The relations (35)-(38) are plainly necessary and it is easy to show that they are sufficient to insure that the system defined by (32) and (33) be of the dynamical type.

12. The following proposition can easily be proved:

The $(n-2)$ -flat associated according to property II with the element $x_1/\omega_1 = x_2/\omega_2 = x_3/\omega_3 = \dots = x_n/\omega_{n-1}$ is at a distance from the point of the element equal to three times the first radius of curvature of the line of force through the point and is perpendicular to the osculating plane at the point of the line of force through it.

THE ASSOCIATED PLANE SYSTEMS S .

13. What we now require is the geometric interpretation of the equations (35)-(38). This is found in connection with certain related systems of plane curves which are termed systems S .*

Consider any plane Π in n -space. At each point of it there are ∞^2 curves of any $(2n-1)$ -fold infinite system of curves in n -space tangent to it. Orthogonal projection upon the plane Π of the differential elements of the third order at the point, going with these curves gives ∞^2 plane differential elements of the third order. In the entire plane there are ∞^4 of these differential elements defining a differential equation of the third order and so determining a system of ∞^3 curves in the plane. This system of curves is termed the associated system S in the plane Π . Since in n -space there are ∞^{3n-6} planes, there are in all ∞^{3n-6} associated systems S .

To determine analytically the S system in the plane Π take the x_1x_2 plane parallel to it and place in the equations of the $(2n-1)$ -fold infinite system of curves in n -space $x_t = C_t$ ($t = 3 \dots n$) where the C_t are constants and $x'_t = 0$ ($t = 3 \dots n$).

14. We are now ready to apply this idea to the dynamical systems in n -space. Since the form of the defining equation is not changed by rotating the axes, it is sufficient to consider the associated system S in a plane parallel to the x_1x_2 plane. Placing $x_t = c_t$ ($t = 3 \dots n$) and $x'_t = 0$ ($t = 3 \dots n$) in the first equation ($j = 2$) of (4) we obtain as the equation of the associated plane system

$$(39) \quad (\bar{\psi}_2 - x_1\bar{\psi}_1)x_2''' = \left| \begin{array}{cc} 1 & \bar{\psi}_{1x_1} + x'_2\bar{\psi}_{1x_2} \\ x'_2 & \bar{\psi}_{2x_1} + x'_2\bar{\psi}_{2x_2} \end{array} \right| x_2'' - 3\bar{\psi}_1x_2''^2$$

where the $\bar{\psi}$ functions are functions of x_1 and x_2 gotten from the ψ functions by the indicated procedure. Since this is exactly the form of the equations of the plane system of trajectories,† we have proved

* For Professor Kasner's discussion of the systems S , see footnote †, page 258.

† See footnote *, page 258.

Property IV. The associated plane systems S connected with the $(2n-1)$ -fold infinite system of dynamical trajectories in n -space are of the two dimensional dynamical type.

15. It is not necessary to state here the geometrical properties of the plane dynamical system as they are given in the paper on Trajectories* already cited.

Denoting them by the same number as in that paper with a subscript p to distinguish them from the n -space properties, we may take as characteristic set: *

$$(40) \quad I_p, II_p, III_p, V_p, VI_p.$$

16. *Converse of Property I_p .* In order that property I_p be true for a system S , its defining equations must have the form

$$(41) \quad x_2''' = G_p x_2'' + H_p x_2'^2$$

where G_p and H_p are arbitrary functions of x_1, x_2 and x_2' . The systems S of our general n -space system are represented by equations of this form as is easily seen from a consideration of equations (16).

17. If the system (41) is to have properties II_p and III_p , G_p and H_p must have the special forms *

$$(42) \quad G_p = \frac{\lambda x_2'^2 + \mu x_2' + \nu}{x_2' - \omega}, \quad H_p = \frac{3}{x_2' - \omega}$$

Consider first the planes parallel to the $x_1 x_2$ plane. Placing $x_t = C_t, x_t' = 0$ ($t = 3 \cdots n$) in

$$(43) \quad G = \frac{\sum_{t=1}^n l_t^{(2)} x_t' - x_2' \sum_{t=2}^n l_t^{(1)} x_t'}{x_2' - \omega_1}, \quad H = \frac{3}{x_2' - \omega_1}$$

$$K_i G + K_i' = \frac{\sum_{t=1}^n l_t^{(1)} x_t' - x_i' \sum_{t=2}^n l_t^{(1)} x_t'}{x_2' - \omega_1} \quad (i = 3 \cdots n).$$

which is the solution of equations (33), and equating the results thus obtained to the corresponding ones from (42), we find

* See footnote *, page 258.

$$(44) \quad \begin{aligned} l_1^{(2)} + l_2^{(2)}x_2' - l_2^{(1)}x_2'^2 &= \lambda x_2'^2 + \mu x_2' + \nu, \quad \omega = \omega_1, \\ l_1^{(i)} + l_2^{(1)}x_2' &= - \\ &\frac{\omega_{i-1}(\lambda x_2'^2 + \mu x_2' + \nu) + (x_2' - \omega_1)\omega'_{i-1} + \omega_{i-1}\omega_1'}{x_2' - \omega_1}, \quad (i = 3 \cdots n). \end{aligned}$$

Substitution of the first of these in the second gives

$$(45) \quad \begin{vmatrix} x_2' - \omega_1 & l_1^{(2)} + \omega_{1x_1} + x_2'(l_2^{(2)} + \omega_{1x_2}) - l_2^{(1)}x_2'^2 \\ -\omega_{i-1} & l_1^{(i)} + \omega_{(i-1)x_1} + x_2'(l_2^{(i)} + \omega_{(i-1)x_2}) \end{vmatrix} = 0 \quad (i = 3 \cdots n).$$

The fact that this is to be satisfied identically shows that

$$(46) \quad \begin{aligned} l_2^{(i)} - \omega_{i-1}l_2^{(1)} + \omega_{(i-1)x_2} &= 0, \\ \omega_1(l_1^{(i)} + \omega_{(i-1)x_1}) - \omega_{i-1}(l_1^{(2)} + \omega_{1x_1}) &= 0 \quad (i = 3 \cdots n) \\ l_1^{(i)} + \omega_{(i-1)x_1} - \omega_1(l_2^{(i)} + \omega_{(i-1)x_2}) + \omega_{i-1}(l_2^{(2)} + \omega_{1x_2}) &= 0. \end{aligned}$$

Now consider in general the planes parallel to the x_1x_j plane. In place of the equation $x_2''' = Gx_2'' + Hx_2'^2$ we might have used

$$(47) \quad x_j''' = \frac{k_j G + k_j'}{k_j} x_j'' + \frac{H}{k_j} x_j'^2.$$

The equations of the systems S in planes parallel to the x_1x_j plane are now obtained by placing in equation (47) $x_i = c_i$, $x'_i = 0$ ($i = 2 \cdots n$, $i \neq j$). Using equations (43) and proceeding as above we obtain

$$(48) \quad \begin{aligned} l_j^{(i)} - \omega_{i-1}l_j^{(i)} + \omega_{(i-1)x_i} &= 0, \\ \omega_{j-1}(l_1^{(i)} + \omega_{(i-1)x_1}) - \omega_{i-1}(l_1^{(j)} + \omega_{(j-1)x_i}) &= 0 \\ &\quad (j = 2 \cdots n, i = 3 \cdots n, i \neq j) \\ l_1^{(i)} + \omega_{(i-1)x_1} - \omega_{j-1}(l_1^{(i)} + \omega_{(i-1)x_j}) + \omega_{i-1}(l_1^{(j)} + \omega_{(j-1)x_j}) &= 0 \end{aligned}$$

The set (48) consists of 3 $(n-1)$ $(n-2)$ equations and these are found to be equivalent to the relations (35). No new relations are found by considering other planes.

18. *Converse of V_p .* By the procedure just used it is easy to show that the relations (35) are also sufficient to insure the truth of *property V_p* .

19. *Converse of VI_p .* The condition for VI_p is

$$(49) \quad \lambda_{x_1} + \left(\frac{\nu + \omega_{x_1}}{\omega} \right)_{x_2} = 0$$

This condition is easily applied to all planes parallel to the x_1x_j ($j = 2 \cdots n$) plane. From equations (43) by placing $x_i = c_i$, $x'_i = 0$, ($i = 2 \cdots n$, $i \neq j$), we obtain

$$(50) \quad \frac{k_j G + k_j'}{k_j} = \frac{l_1^{(j)} + l_j^{(j)} x_j' - l_j^{(1)} x_j'^2}{x_j' - \omega_{j-1}}, \quad \frac{H}{k_j} = \frac{3}{x_j' - \omega_{j-1}},$$

($j = 2 \cdots n$).

From these it is seen that

$$(51) \quad \lambda = -l_j^{(1)}, \quad \mu = l_j^{(j)}, \quad \nu = l_1^{(j)}, \quad \omega = \omega_{j-1}.$$

When these values are substituted in equation (49) it becomes

$$(52) \quad -l_{jx_1}^{(1)} + \left(\frac{l_1^{(j)} + \omega_{(j-1)} x_1}{\omega_{j-1}} \right)_{x_j} = 0 \quad (j = 2 \cdots n).$$

These are the relations (37).

In order to consider planes parallel to the $x_i x_j$ plane, it is necessary to transform our general equations so that x_j shall be the independent variable. The result of this transformation is contained in equations (57) and (57'). The intervening discussion suggests how the transformation is carried out.

In the remaining part of this article primes shall denote derivatives with respect to x_j , those with respect to x_1 being written out in full.

The equation most easily transformed is

$$(53) \quad \frac{d^3 x_j}{dx_1^3} = \frac{K_j G + \frac{dK_j}{dx_1}}{K_j} \frac{d^2 x_j}{dx_1^2} + \frac{H}{K_j} \left(\frac{d^2 x_j}{dx_1^2} \right)^2 \quad (j = 2 \cdots n).$$

Transformation of this so that x_j becomes the independent variable gives

$$(54) \quad x_1''' = \frac{K_j G + \frac{dK_j}{dx_1}}{K_j} x_1' \cdot x_1'' - \frac{\frac{H}{K_j} - 3x_1'}{x_1'^2} (x_1'')^2.$$

From this it is easy to derive the equation,

$$(55) \quad x_i''' = \frac{k_i \left(K_j G + \frac{dK_j}{dx_1} \right) x_1' + k_i'}{k_i} x_i'' - \frac{\frac{H}{K_j} - 3x_1'}{x_1' k_i} x_i''^2$$

where

$$(55') \quad x_i'' = \frac{x_i' - x_1' \omega_{i-1}}{x_1' - x_1' \omega_{j-1}} x_1'' = k_i x_1'' \quad (j = 2 \cdots n, i = j+1, \cdots n).$$

Substituting in equation (55) the value of $K_j G + dK_j/dx_1$ from (43) and using the following equation:

$$(56) \quad -k_i' = \frac{d}{dx_1} \left(\frac{K_i}{K_j} \right) \\ = \frac{\sum_{t=1}^n l_t^{(i)} x_t' - x_i' / x_1' \sum_{t=2}^n l_t^{(1)} x_t' - \frac{x_i' - x_1' \omega_{i-1}}{1 - x_1' \omega_{j-1}} \left[\sum_{t=1}^n l_t^{(j)} x_t' - 1/x_1' \sum_{t=2}^n l_t^{(1)} x_t' \right]}{1 - \omega_{j-1} x_1'} \quad (i = 1 \cdots n, i \neq j),$$

we find

$$(57) \quad x_i''' = \frac{\lambda x_i'^2 + \mu x_i' + \nu}{x_i' - \omega} x_i'' + \frac{3}{x_2' - \omega} x_i''^2$$

where

$$(57') \quad \lambda = \frac{l_i^{(j)}}{\omega_{j-1}}, \quad \mu = \frac{l_i^{(i)} - \left[\sum_{t=1}^n l_t^{(j)} x_t' (t \neq i) \right]}{\omega_{j-1}}, \quad \nu = \frac{\sum_{t=1}^n l_t^{(i)} x_t' (t \neq i)}{\omega_{j-1}}, \\ \omega = \frac{\omega_{i-1}}{\omega_{j-1}}.$$

Substituting in the equations (57') $x_t = c_t$, $x_t' = 0$ ($t = 1 \cdots n$, $t \neq i$ or j), placing the result in (49) and simplifying by (35) we obtain

$$(58)^* \quad \left(\frac{-l_i^{(j)}}{\omega_{j-1}} \right)_{x_j} + \left[\frac{l_j^{(i)}}{\omega_{j-1}} + \left(\frac{\omega_{i-1}}{\omega_{j-1}} \right)_{x_j} \right]_{x_i} = \left(\frac{-l_i^{(j)}}{\omega_{j-1}} \right)_{x_j} - \left(\frac{\omega_{(j-1)} x_j}{\omega_{j-1}} \right)_{x_i} \\ + \left(\frac{l_j^{(i)} + \omega_{(i-1)} x_j}{\omega_{i-1}} \right)_{x_i} = - \left(\frac{l_j^{(i)} + \omega_{(i-1)} x_i}{\omega_{j-1}} \right)_{x_j} + \left(\frac{l_i^{(j)} + \omega_{(j-1)} x_i}{\omega_{i-1}} \right)_{x_i} \\ = -l_{ix_j} + l_{jx_i} = 0.$$

These are the equations (38).

Already we have a system which fulfils the conditions imposed by equations (34). Accordingly, on account of property IV, other planes can give rise to no new relations.

In order that a $(2n-1)$ -fold infinite system of curves defined by the equations (32)-(32') and (33) therefore possessing properties I, II and III, also possess property IV, it is necessary and sufficient that it be identifiable with the system of n -space trajectories due to a positional field of force.

20. The characterizing properties then are as follows:

I. *The osculating planes at a given point of all curves of the system through that point form a hyperpencil of planes in n -space, that is, all osculating planes at a point pass through a fixed line.*

* When $j = 1$ in formula (58), equations (37) result if $\omega_0 = 1$.

II. The centers of the third order osculating hyperspheres at a point of the ∞^1 curves passing through that point in a given direction lie in an $(n-2)$ -flat.

III. The ∞^{n-1} $(n-2)$ -flats which correspond according to property II to the ∞^{n-1} elements at a point make up the secant $(n-2)$ -flats of an n -th order n -space curve which passes through the point in the direction of the axis of the hyperpencil of osculating planes at the point.

IV. The associated plane systems S (described in article (13) connected with the $(2n-1)$ -fold infinite system of curves in n -space are of the two-dimensional dynamical type.

No one of these properties can be derived from those preceding it. Property IV may be replaced by weaker conditions. It is sufficient that the plane systems S involved in it, in $n(n-1)/2$ orthogonal systems of parallel planes, possess the properties II_p , III_p , and VI_p .

Rods of Constant or Variable Circular Cross Section.*

BY CARL A. GARABEDIAN.

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PART I. THE THEORY.

§ 1. *Introduction.* In a paper on "Circular Plates of Constant or Variable Thickness," † the writer developed in detail a method of series in elasticity and called attention to the possibilities of extension of the method. Having dealt with problems in circular plates, it seems natural that we should turn next to rods and inquire if the method is not available for expansions in the "neighborhood of an axis" as well as for expansions in the "neighborhood of a plane."

Cylindrical coördinates are adopted, and the rod is generated by revolution of a curve

$$1) \quad r = f(z)$$

* Presented to the American Mathematical Society, April 28, 1923. See also *Comptes Rendus*, Vol. 177, Nov. 12, 1923, p. 942.

† *Trans. Amer. Math. Soc.*, Vol. 25, No. 3, p. 343, July, 1923.

about the axis of z . Let the upper and lower ends be two plane sections, and for convenience denote them by:

$$2a) \quad z = h_1,$$

$$2b) \quad z = h_2,$$

respectively.

We restrict ourselves to problems in which the displacements take place in planes through the axis and are the same in all such planes; that is, the ends of the rod may be under tension (or pressure, if the rod is not too long), and we admit applied tractions (independent of θ) everywhere normal to the surface of the rod; but bending or twisting is excluded. The chief interest in this class of problems lies in the fact that the radius of the section may vary with z .

In the discussion that follows, the paper on circular plates will be referred to as paper I, and details that were discussed at length in that paper will be passed over briefly in the sequel.

Introduce a new variable ρ such that:

$$3) \quad r = \rho\tau,$$

and at the same time write:

$$4) \quad f(z) = \phi(z)\tau.$$

The surface of the rod may then be written in either of the forms:

$$5a) \quad r = \phi\tau,$$

$$5b) \quad \rho = \phi.$$

Our *fundamental assumption* is that quantities which are functions of r and z can be expanded in ascending positive integral powers of τ —in particular that:

$$6a) \quad U(r, z) = U_0(\rho, z) + U_1(\rho, z)\tau + U_2(\rho, z)\tau^2 + \dots,$$

$$6b) \quad w(r, z) = w_0(\rho, z) + w_1(\rho, z)\tau + w_2(\rho, z)\tau^2 + \dots,$$

where U and w denote the radial and axial displacements respectively.

If we write $f(z) = r_0 F(z)$, $\phi(z) = \rho_0 F(z)$, where $r_0 = \rho_0 \tau$ is a constant multiplier, it turns out that the coefficient of τ^n in 6a) or 6b) is a homogeneous polynomial in ρ and ρ_0 of degree n . When the return is made to the original variable—by setting $\tau = 1$ and replacing ρ by r —the terms in 6) are homogeneous polynomials in r and r_0 and are ordered according to degree. The coefficients in these polynomials are functions of z , and in general involve F and its derivatives, the elastic constants, the applied tractions, and the

constants h_1, h_2 . Of chief concern is the fact that the manner of occurrence of F and its derivatives is subject to the requirement of homogeneity in r and r_0 .

We take the point of view of pressures prescribed on the surface of the rod, and compute the displacements by determining the coefficients in 6) in such fashion that:

- i) the body forces have the required values,
- ii) the surface tractions are as prescribed,
- iii) the boundary conditions at the ends are satisfied.

In the next paragraph these requirements are expressed as identities in τ .

The notation adopted is that of paper I; in particular, we use the "star" operator defined by:

$$7) \quad A^* = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A) = A' + \frac{A}{\rho},$$

the accent denoting differentiation with regard to ρ .

§ 2. *The body force and surface traction conditions.* Using 3), and taking the body forces F_r, F_z to be nil, the stress equations of equilibrium—or "body force conditions"—become:

$$8a) \quad \frac{2(1-\sigma)}{\tau^2} U^{*'} + \frac{1}{\tau} \frac{\partial w'}{\partial z} + (1-2\sigma) \frac{\partial^2 U}{\partial z^2} \equiv 0,$$

$$8b) \quad 2(1-\sigma) \frac{\partial^2 w}{\partial z^2} + \frac{1}{\tau} \frac{\partial U^*}{\partial z} + \frac{1-2\sigma}{\tau^2} w'^* \equiv 0.$$

To derive conditions for tractions applied to the surface of the rod, we proceed as in paper I and find: †

$$9a) \quad \left. \widehat{r\nu} \right|_{r=f} = \frac{\widehat{rr} - \widehat{zr\tau\phi'}}{\sqrt{1 + \tau^2\phi'^2}} \Big|_{\rho=\phi} \equiv \frac{R}{\tau \sqrt{1 + \tau^2\phi'^2}},$$

$$9b) \quad \left. \widehat{z\nu} \right|_{r=f} = \frac{\widehat{zr} - \widehat{zz\tau\phi'}}{\sqrt{1 + \tau^2\phi'^2}} \Big|_{\rho=\phi} \equiv \frac{-\tau\phi'R}{\tau \sqrt{1 + \tau^2\phi'^2}}.$$

Here we denote the normal traction by:

$$10) \quad R = R_0 + R_1\tau + R_2\tau^2 + \dots,$$

the coefficients being functions of z only. Indeed, we have set $\rho = \phi(z) = \rho_0 F(z)$, and hence 10) is a power series in $\rho_0\tau$ —that is, in r_0 .

† In the case of ϕ , the accent denotes differentiation with regard to z .

Using the elastic constants E and σ , and the relations ~~connecting~~ stress-components with displacements, we may write the **identities 9)** in the form:

$$11a) \quad \frac{2\sigma}{\tau} \frac{U}{\rho} + 2\sigma \frac{\partial w}{\partial z} + \frac{2(1-\sigma)}{\tau} U' - (1-2\sigma) \left[\frac{\partial U}{\partial z} \tau + w' \right] \phi' \Big|_{\rho=\phi} \\ \equiv \frac{2(1+\sigma)(1-2\sigma)R}{E},$$

$$11b) \quad (1-2\sigma) \left[\frac{\partial U}{\partial z} + \frac{1}{\tau} w' \right] - 2\sigma U' \phi' - 2(1-\sigma) \frac{\partial w}{\partial z} \tau \phi' \Big|_{\rho=\phi} \\ \equiv - \frac{2\tau \phi' (1+\sigma)(1-2\sigma)R}{E}.$$

In prescribing a displacement or traction † at an end, conventional use must be made of Saint-Venant's principle. For example, we might have a first boundary condition like $w(0,0) = 0$, together with a second boundary condition of the form:

$$12) \quad \int_0^{2\pi} \int_0^{\phi(h)} \widehat{zz} \Big|_{z=h} \tau^2 \rho d\rho d\theta = Z_h,$$

where Z_h is a *total* pressure or tension at the end, and

$$13) \quad Z_h = Z_0 + Z_1\tau + Z_2\tau^2 + \dots$$

In terms of the displacements and elastic constants E and σ , relation 12) becomes:

$$14) \quad \frac{2\pi E}{(1+\sigma)(1-2\sigma)} \int_0^{\phi(h)} \left[\sigma U' \tau + (1-\sigma) \frac{\partial w}{\partial z} \tau^2 \right]_{z=h} \rho d\rho \equiv Z_h.$$

Since the bracketed portion of the integrand may involve ρ , we cannot integrate with regard to that variable at the present stage of the work.

The sense is such that R , Z_h are tensions when positive, pressures when negative (Cf. Paper I).

§ 3. *Analysis of the formulas of displacement by the aid of certain transformations.* In paper I, we found it an advantage to introduce certain transformations as an aid in shaping the theory. In the case of rods, the useful transformations to consider are the following:

$$15a) \quad (\tau, -\tau), (U, -U),$$

$$15b) \quad (\rho, -\rho), (U, -U), (\phi, -\phi),$$

† Since we are concerned with the *total* applied traction at an end, we may admit certain cases where the actual distribution of traction depends upon θ as well as r .

where $(\tau, -\tau)$ signifies that τ is replaced by $-\tau$, etc. The three sets of identities in τ of the preceding paragraph are left unaltered by the transformation 15b); and if

$$16) \quad R(\tau) = R(-\tau), \quad Z_h(\tau) = Z_h(-\tau),$$

they are also invariant under the transformation 15a).

When the displacements are prescribed, we have $R \equiv R_0$ and $Z_h \equiv Z_2 \tau^2$, and the conditions 16) are always fulfilled; hence we may immediately apply both transformations to the simplification of the formulas of displacement. It is readily seen that 15a) and 15b) require respectively that:

$$17a) \quad U_{2m}(\rho, z) \equiv w_{2m+1}(\rho, z) \equiv 0, \quad m = 0, 1, 2, \dots,$$

$$17b) \quad U_m(\rho, z) + U_m(-\rho, z) \equiv w_m(\rho, z) - w_m(-\rho, z) \equiv 0, \\ m = 0, 1, 2, \dots$$

We thus see that the radial displacement contains only odd powers of τ , and that the axial displacement contains only even powers of τ . Furthermore, the coefficients that remain in the radial displacement are odd functions of ρ , and the coefficients that remain in the axial displacement are even functions of ρ . With this important information available, the computation of the displacements is considerably simplified.

§ 4. *Determination of formulas of displacement that satisfy the body force conditions.* By setting equal to 0 the coefficients of τ^n in 8a) and 8b), we obtain equations which we denote by $F_r^{(n)} = 0$, $F_z^{(n)} = 0$, respectively. Since coefficients in 6a) vanish when the subscript is even, we are interested in $F_r^{(n)} = 0$ only for n odd; similarly, we compute $F_z^{(n)} = 0$ only for n even.

The equation $F_z^{(-2)} = 0$ is:

$$18a) \quad w_0'^* = 0;$$

and integrating, we have:

$$18b) \quad w_0 = w_{0a}(z) \log \rho + w_{0b}(z),$$

where w_{0a} , w_{0b} are arbitrary functions of z . Since $\log \rho$ becomes infinite on the axis of z , we suppress this term; that is, we set $w_{0a} \equiv 0$ and write:

$$18c) \quad w_0 = w_{0b}(z).$$

For $F_r^{(-1)} = 0$ we find:

$$19a) \quad 2(1 - \sigma) U_1^{*'} + \frac{\partial w_0'}{\partial z} = 0;$$

but $w_0' = w_{0b}' = 0$, and the equation to be solved is:

$$19b) \quad U_1^{**} = 0.$$

We are thus led to:

$$19c) \quad U_1 = U_{1a}(z)\rho + \frac{U_{1b}(z)}{\rho};$$

and since the term in ρ^{-1} becomes infinite on the axis of the rod, we set $U_{1b} \equiv 0$ and write:

$$19d) \quad U_1 = U_{1a}(z)\rho.$$

We turn next to $F_z^{(0)} = 0$. Integrating this equation and suppressing the term in $\log \rho$, we find:

$$20) \quad w_2 = w_{2b} - \frac{\rho^2}{2(1-2\sigma)} \left[\frac{dU_{1a}}{dz} + (1-\sigma) \frac{d^2w_{0b}}{dz^2} \right].$$

After integrating $F_r^{(1)} = 0$, we suppress the term in ρ^{-1} and obtain:

$$21) \quad U_3 = U_{3a}\rho + \frac{\rho^3}{16(1-2\sigma)} \left[4\sigma \frac{d^2U_{1a}}{dz^2} + \frac{d^3w_{0b}}{dz^3} \right].$$

It is convenient to have also the formulas for w_4 and U_5 —obtained from $F_z^{(2)} = 0$, $F_r^{(3)} = 0$, respectively. We find:

$$22a) \quad w_4 = w_{4b} - \frac{\rho^2}{2(1-2\sigma)} \frac{dU_{3a}}{dz} - \frac{(1-\sigma)\rho^2}{2(1-2\sigma)} \frac{d^2w_{2b}}{dz^2} \\ + \frac{\rho^4}{16(1-2\sigma)} \frac{d^3U_{1a}}{dz^3} + \frac{(3-2\sigma)\rho^4}{64(1-2\sigma)} \frac{d^4w_{0b}}{dz^4},$$

$$22b) \quad U_5 = U_{5a}\rho + \frac{\rho^3}{4(1-2\sigma)} \frac{d^2U_{3a}}{dz^2} + \frac{\rho^3}{16(1-2\sigma)} \frac{d^3w_{2b}}{dz^3} \\ - \frac{(1+2\sigma)\rho^5}{192(1-2\sigma)} \frac{d^4U_{1a}}{dz^4} - \frac{\rho^5}{192(1-2\sigma)} \frac{d^5w_{0b}}{dz^5}.$$

By means of the formulas obtained in this paragraph, the displacements:

$$23a) \quad U = U_1\tau + U_3\tau^3 + U_5\tau^5 + \dots,$$

$$23b) \quad w = w_0 + w_2\tau^2 + w_4\tau^4 + \dots,$$

correspond to a body force whose radial and axial components vanish up to orders τ^5 and τ^4 , respectively.

§ 5. *Differential equations obtained from the surface traction conditions.* We denote by $R^{(n)} = 0$, $R_{(n)} = 0$, respectively, the equations obtained from 11a) and 11b) by equating coefficients of τ^n . These equations are total linear

differential equations, and they determine, up to constants of integration, the arbitrary functions of z of § 4. To be explicit, the equations

$$24a) \quad R^{(2m)} = 0, \quad R_{(2m+1)} = 0$$

determine the unknowns

$$24b) \quad w_{2m,b} \text{ and } U_{2m+1,a}$$

where $m = 0, 1, 2, \dots$. We shall compute these differential equations for $m = 0, 1$.

For $R^{(0)} = 0$ we find:

$$25a) \quad U_{1a} = \frac{(1+\sigma)(1-2\sigma)R_0}{E} - \sigma \frac{dw_{0b}}{dz},$$

and for $R_{(1)} = 0$:

$$25b) \quad (1-\sigma)\phi \frac{d^2 w_{0b}}{dz^2} + 2\sigma\phi \frac{dU_{1a}}{dz} + 2(1-\sigma)\phi' \frac{dw_{0b}}{dz} + 4\sigma\phi' U_{1a} \\ = \frac{2(1+\sigma)(1-2\sigma)\phi'R_0}{E}.$$

It is natural to eliminate U_{1a} from 25b). The result is the following differential equation for the determination of w_{0b} :

$$26) \quad \frac{d^2 w_{0b}}{dz^2} + 2 \frac{\phi'}{\phi} \frac{dw_{0b}}{dz} = \frac{2(1-2\sigma)\phi'R_0}{E\phi} - \frac{2\sigma}{E} \frac{dR_0}{dz}.$$

When w_{0b} has been found, we may return to 25a) and compute U_{1a} .

When we write out in full the equations $R^{(2)} = 0$, $R_{(3)} = 0$, the result is:

$$27a) \quad U_{3a} = \frac{(1+\sigma)(1-2\sigma)R_2}{E} - \sigma \frac{dw_{2b}}{dz} - \frac{3-4\sigma}{16} \phi^2 \frac{d^3 w_{0b}}{dz^3} \\ - \frac{\sigma}{4} \phi^2 \frac{d^2 U_{1a}}{dz^2} - \frac{1-\sigma}{2} \phi\phi' \frac{d^2 w_{0b}}{dz^2} - \sigma\phi\phi' \frac{dU_{1a}}{dz},$$

$$27b) \quad (1-\sigma)\phi \frac{d^2 w_{2b}}{dz^2} + 2\sigma\phi \frac{dU_{3a}}{dz} + 2(1-\sigma)\phi' \frac{dw_{2b}}{dz} + 4\sigma\phi' U_{3a} \\ = \frac{2(1+\sigma)(1-2\sigma)\phi'R_2}{E} + \phi'\phi^2 \left[(1+\sigma) \frac{d^2 U_{1a}}{dz^2} + \frac{2-\sigma}{2} \frac{d^3 w_{0b}}{dz^3} \right] \\ + \frac{\phi^3}{4} \left[(1+\sigma) \frac{d^3 U_{1a}}{dz^3} + \frac{2-\sigma}{2} \frac{d^4 w_{0b}}{dz^4} \right].$$

We know ϕ , w_{0b} and U_{1a} have already been determined, and, since we are prescribing the traction at the surface, $R_2 = 0$. Hence equations 27) determine w_{2b} and U_{3a} . Eliminating U_{3a} from 27b), we obtain for the determination of w_{2b} a differential equation with the same reduced linear homogeneous

equation † as 26). So if we can integrate 26), we expect in general to be able to integrate subsequent equations. Solutions of the differential equations of this paragraph for the case $R_0 = \text{const.}$ are obtained in Part II of the paper, in connection with the applications.

§ 6. *Boundary conditions at the ends of a rod.* By 26), the function w_{0b} will contain two constants of integration. One of these constants is determined by the assignment of an axial displacement that shall correspond to a given z ; for example, it may be convenient to demand that $w(0, 0) = 0$, which implies $w_{0b} = 0$, $w_{2b} = 0$, etc. The remaining constant may be determined by assigning the axial displacement at some new value of z , or it may be determined by a condition of the form 14). In applying the latter type of boundary condition, we need to know the relations obtained from 14) by equating coefficients of like powers of τ ; Z_h is given by 13), and U and w have the values computed in §§ 4, 5.

We find $Z_0 \equiv 0$, and the terms in τ^2 and τ^4 yield respectively the relations:

$$28) \quad \frac{E}{(1+\sigma)(1-2\sigma)} \left[2\sigma U_{1a} + (1-\sigma) \frac{dw_{0b}}{dz} \right]_{z=h} = P_0,$$

$$29) \quad \frac{E}{(1+\sigma)(1-2\sigma)} \left[2\sigma U_{3a} + (1-\sigma) \frac{dw_{2b}}{dz} - \frac{1+\sigma}{4} \phi^2 \frac{d^2 U_{1a}}{dz^2} - \frac{2-\sigma}{8} \phi^2 \frac{d^2 w_{0b}}{dz^2} \right]_{z=h} = 0,$$

where (to simplify the applications) we have assumed a *constant* unit traction of amount P_0 ; that is, we have:

$$30) \quad Z_h \equiv Z_2 \tau^2 = \pi \phi^2(h) \tau^2 P_0.$$

In general, the unit traction on the end is a function of both r and θ (Cf. footnote, § 2).

PART II. APPLICATIONS.

§ 7. *Rod of uniform cross section.* When the section is of constant radius, the rod becomes a right circular cylinder. This very simple case is not without interest; indeed, we have here the possibility of solutions that are not only "exact" but also without restriction upon the thinness of the rod. As examples of this sort, we shall consider two problems which were solved in paper I. It is curious that we should be able thus to establish contact

† Cf. § 8.

between the method of series as applied to plates and the method of series as applied to rods.

When the rod is of uniform section ($\phi' = 0$), 26) reduces to:

$$31a) \quad \frac{d^2 w_{0b}}{dz^2} = 0.$$

We assume R_0 to be a uniform pressure of amount p_0 ; that is, we replace R_0 by the constant $-p_0$. Integrating 31a), we have:

$$31b) \quad w_{0b} = C_1 z + C_2,$$

Also, from 25a) there results:

$$32) \quad U_{1a} = -\frac{(1-\sigma)(1-2\sigma)p_0}{E} - \sigma C_1.$$

We assume throughout this paragraph that the axial displacement at $z = 0$ is nil; that is, $C_2 = 0$. Also, we take the ends of the rod to be the planes $z = \pm h$; and in view of the complete symmetry in the plane $z = 0$, it will suffice to consider the end $z = h$.

Let us consider first the case in which there is no traction on the ends; that is $Z_2 \equiv 0$. Then 28) gives:

$$33a) \quad -2\sigma \left[\frac{(1+\sigma)(1-2\sigma)p_0}{E} + \sigma C_1 \right] + (1-\sigma)C_1 = 0,$$

or:

$$33b) \quad C_1 = \frac{2\sigma p_0}{E},$$

and we have:

$$34a) \quad w_{0b} = \frac{2\sigma p_0 z}{E}$$

$$34b) \quad U_{1a} = -\frac{(1+\sigma)(1-2\sigma)p_0}{E} - \frac{2\sigma^2 p_0}{E}.$$

It is readily seen that all remaining unknowns vanish identically. For example, we find:

$$35a) \quad w_{2b} = C_1' z,$$

$$35b) \quad U_{3a} = -\sigma C_1',$$

and the condition 29) gives $C_1' = 0$; that is, $w_{2b} \equiv U_{3a} \equiv 0$.

By 23), the final formulas of displacement are:

$$36a) \quad U = -\frac{(1-\sigma)p_0 r}{E},$$

$$36b) \quad w = \frac{2\sigma p_0 z}{E}.$$

These results agree with those obtained by setting $r_1 = 0$ in 75), paper I.

Secondly, suppose that we keep the length of the rod constant by imposing a pressure at the ends. We have:

$$37) \quad w_{0b}(h) = C_1 h = 0,$$

or $C_1 = 0$; that is, $w_{0b} \equiv 0$. In this case, we compute the traction at the ends by means of 28), and find:

$$38) \quad P_0 = -2\sigma p_0;$$

and this result is in agreement with the relation $S_0^0 = -2\sigma p_0$ obtained from 81a) of paper I.

Since all functions vanish identically save

$$39) \quad U_{1a} = - \frac{(1 + \sigma)(1 - 2\sigma)p_0}{E},$$

the formulas of displacement are:

$$40a) \quad U = - \frac{(1 + \sigma)(1 - 2\sigma)p_0 r}{E},$$

$$40b) \quad w = 0;$$

that is, we have the solution obtained by setting $r_1 = 0$ in 80), paper I.

§ 8. *Rods of variable cross section under uniform pressure.* In the general case of variable cross section, the first equation to be integrated is 26). Let us restrict R_0 (as in the preceding paragraph) to be a constant pressure: $R_0 = -p_0$. Equation 26) reduces to:

$$41) \quad \frac{d^2 w_{0b}}{dz^2} + 2 \frac{\phi'}{\phi} \frac{dw_{0b}}{dz} = - \frac{2(1 - 2\sigma)p_0}{E} \frac{\phi'}{\phi}.$$

If we write:

$$42a) \quad y = \frac{dw_{0b}}{dz},$$

$$42b) \quad K = - \frac{(1 - 2\sigma)p_0}{E},$$

equation 41) becomes:

$$43) \quad \frac{dy}{dz} + 2 \frac{\phi'}{\phi} y = 2 \frac{\phi'}{\phi} K.$$

This equation has the explicit solution:

$$44) \quad y = K + \frac{C_1}{\phi^2},$$

and we can now write w_{0b} in terms of a single quadrature. We have:

$$45a) \quad w_{0b} = \int \left[K + \frac{C_1}{\phi^2} \right] dz + C_2,$$

$$45b) \quad U_{1a} = K - \frac{\sigma C_1}{\phi^2}.$$

After integrating 43) explicitly, it is natural to inquire next if the differential equation which determines w_{2b} is not capable of similar treatment. Eliminating U_{3a} from equations 27), we find:

$$46) \quad \frac{d^2 w_{2b}}{dz^2} + 2 \frac{\phi'}{\phi} \frac{dw_{2b}}{dz} = 2C_1(5 - 3\sigma) \frac{\phi' \phi''}{\phi^2} - \frac{(1 + \sigma)C_1}{2} \frac{\phi'''}{\phi}.$$

Using 42a), we have a linear equation which differs from 43) only in the respect that the non-homogeneous member is more involved. However, the equation has the explicit solution:

$$47) \quad y = \frac{dw_{2b}}{dx} = \frac{1}{\phi^2} \left[\frac{(21 - 11\sigma)C_1}{4} (\phi')^2 - \frac{(1 + \sigma)C_1}{2} \phi \phi'' + C_1' \right],$$

and w_{2b} , as well as w_{0b} , is expressible in terms of a single quadrature. The final results are:

$$48a) \quad w_{2b} = \int \frac{1}{\phi^2} \left[\frac{(21 - 11\sigma)C_1}{4} (\phi')^2 - \frac{(1 + \sigma)C_1}{2} \phi \phi'' + C_1' \right] dz + C_2',$$

$$48b) \quad U_{3a} = \frac{1}{\phi^2} \left[\frac{(18\sigma^2 - 38\sigma - 1)C_1}{8} (\phi')^2 + \frac{3C_1}{8} \phi \phi'' - C_1' \right].$$

It is apparently possible to continue to integrate explicitly the linear equations that present themselves, and thus to express all subsequent functions in terms of a single quadrature at most.

From 45) and 48), we can compute leading term in the formulas of displacement. By way of concrete illustration, we proceed to give two examples.

§ 9. *Uniformly tapering rod.* As equation of the uniformly tapering rod, it is convenient to write:

$$49a) \quad f(z) = r_0(mz + b),$$

or:

$$49b) \quad \phi(z) = \rho_0(mz + b),$$

where (as agreed) $f = \phi t$, $r_0 = \rho_0 \tau$. For the particular ϕ in question, the formulas 45) and 48) reduce to:

$$50a) \quad w_{0b} = Kz - \frac{C_1}{m(mz+b)\rho_0^2} + C_2,$$

$$50b) \quad U_{1a} = K - \frac{\sigma C_1}{(mz+b)^2 \rho_0^2},$$

$$50c) \quad w_{2b} = -\frac{1}{m(mz+b)\rho_0^2} \left[\frac{(21-11\sigma)C_1 m^2 \rho_0^2}{4} + C_1' \right] + C_2',$$

$$50d) \quad U_{3a} = \frac{1}{(mz+b)^2 \rho_0^2} \left[\frac{(18\sigma^2 - 38\sigma - 1)C_1 m^2 \rho_0^2}{8} - \sigma C_1' \right].$$

For the sake of a definite problem that will determine the constants of integration, let us demand first that $w(0,0) = 0$, and secondly that the upper end ($z = h$) be subject to unit tension of amount P_0 :

$$51) \quad Z_2 = \pi \rho_0^2 (mh + b)^2 P_0.$$

The first of these boundary conditions requires that $w_{0b}(0) = w_{2b}(0) = 0$, and determines C_2 and C_2' in terms of C_1 and C_1' , respectively. By means of relations 28) and 29), the second boundary condition determines C_1 and C_1' . The functions 50) being known, it remains to compute the displacements from the formulas of § 4. We find (making the return to the original variable r):

$$52a) \quad U = -\frac{1}{E} \left[(1-2\sigma)p_0 + \sigma(p_0 + P_0) \left(\frac{mh+b}{mz+b} \right)^2 \right] r \\ - \frac{p_0 + P_0}{8E} \left(\frac{mh+b}{mz+b} \right)^2 \left[(1+8\sigma)m^2 r_0^2 r - \frac{3(1+2\sigma)m^2}{(mz+b)^2} r^3 \right] + \dots,$$

$$52b) \quad w = \frac{1}{E} \left[\frac{(p_0 + P_0)(mh+b)^2 z}{b(mz+b)} - 1(-2\sigma)p_0 z \right] \\ + \frac{p_0 + P_0}{E} \left[\frac{(3-\sigma)m^2(mh+b)^2 z}{2b(mz+b)} r_0^2 + \frac{m(mh+b)^2}{(mz+b)^3} r^2 \right] + \dots$$

Recall that p_0 is the pressure on the surface of the rod, and P_0 the tension on the end $z = h$. The r_0 serves merely to bring out the homogeneity (in r and r_0) of the bracketed terms; it may be suppressed if desired.

When $m = 0$ and $P_0 = 0$, 52) reduces to 36). If $m = 0$ and $P_0 = -2\sigma p_0$, 52) reduces to 40). We have thought of the ends of the rod as the planes $z = h$ and $z = 0$, but for purposes of comparison with 36) and 40), we interpret our formulas for a rod whose lower end is $z = -h$.

§ 10. *A bulging rod.* In this paragraph we consider the rod generated when the curve

$$53a) \quad r = \frac{r_0}{a^2 + z^2}$$

revolves about the axis of z ; that is, we have:

$$53b) \quad \phi = \frac{\rho_0}{a^2 + z^2}.$$

From 45) and 48), we find:

$$54a) \quad w_{0b} = Kz + \frac{C_1}{\rho_0^2} \left[a^4 z + \frac{2}{3} a^2 z^3 + \frac{1}{5} z^5 \right] + C_2,$$

$$54b) \quad U_{1a} = K - \frac{\sigma C_1}{\rho_0^2} (a^2 + z^2)^2,$$

$$54c) \quad w_{2b} = \frac{(19 - 13\sigma)C_1}{2a} \tan^{-1} \frac{z}{a} - \frac{(17 - 15\sigma)C_1 z}{2(a^2 + z^2)} + \frac{C_1'}{\rho_0^2} \left[a^4 z + \frac{2}{3} a^2 z^3 + \frac{1}{5} z^5 \right] + C_2',$$

$$54d) \quad U_{3a} = \frac{(18\sigma^2 - 38\sigma + 5)C_1 z^2}{2(a^2 + z^2)^2} - \frac{3C_1}{4(a^2 + z^2)} - \frac{\sigma C_1 (a^2 + z^2)^2}{\rho_0^2}.$$

Imposing the condition $w(0, 0) = 0$, we have $C_2 = C_2' = 0$. We further demand a unit tension of amount P_0 at $z = h$; and in view of the symmetry in $z = 0$, this same condition obtains at $z = -h$. The relations 54) become:

$$55a) \quad w_{0b} = \frac{\rho_0 + P_0}{E(a^2 + h^2)^2} \left[a^4 z + \frac{2}{3} a^2 z^3 + \frac{1}{5} z^5 \right] - \frac{(1 - 2\sigma)\rho_0 z}{E}$$

$$55b) \quad U_{1a} = -\frac{\sigma(\rho_0 + P_0)}{E(a^2 + h^2)^2} (a^2 + z^2)^2 - \frac{(1 - 2\sigma)\rho_0}{E},$$

$$55c) \quad w_{2b} = \frac{\rho_0^2(\rho_0 + P_0)}{E(a^2 + h^2)^2} \left\{ \frac{19 - 13\sigma}{2a} \tan^{-1} \frac{z}{a} - \frac{(17 - 15\sigma)z}{2(a^2 + z^2)} + \frac{3(3\sigma - 5)z^2}{(a^2 + z^2)^4} \left[a^4 z + \frac{2}{3} a^2 z^3 + \frac{1}{5} z^5 \right] \right\},$$

$$55d) \quad U_{3a} = \frac{\rho_0^2(\rho_0 + P_0)}{4E(a^2 + h^2)^2(a^2 + z^2)^2} \left[(7 - 16\sigma)z^2 - 3a^2 \right].$$

The formulas of § 4 give the following displacements:

$$\begin{aligned}
 56a) \quad U &= -\frac{1}{E} \left[(1-2\sigma)p_0 + \sigma(p_0 + P_0) \left(\frac{a^2 + z^2}{a^2 + h^2} \right)^2 \right] r \\
 &+ \frac{(p_0 + P_0)}{4E(a^2 + h^2)^2} \left[\frac{(7-16\sigma)z^2 - 3a^2}{(a^2 + z^2)^2} r_0^2 r + (1+2\sigma)(a^2 + 3z^2)r^3 \right] \\
 &\quad + \dots, \\
 56b) \quad w &= \frac{1}{E} \left[\frac{p_0 + P_0}{(a^2 + h^2)^2} \left\{ a^4 z + \frac{2}{3} a^2 z^3 + \frac{1}{5} z^5 \right\} - (1-2\sigma)p_0 z \right] \\
 &\quad + \frac{p_0 + P_0}{2E(a^2 + h^2)^2} \left[\left\{ \frac{19-13\sigma}{a} \tan^{-1} \frac{z}{a} - \frac{(17-15\sigma)z}{a^2 + z^2} \right. \right. \\
 &\quad \left. \left. + \frac{6(3\sigma-5)z^2}{(a^2 + z^2)^4} (a^4 z + \frac{2}{3} a^2 z^3 + \frac{1}{5} z^5) \right\} r_0^2 \right. \\
 &\quad \left. - 4z(a^2 + z^2)r^2 \right] + \dots.
 \end{aligned}$$

§ 11. *Critique of method.*† For the right circular cylinder, the series for the displacements terminate, and when the parameter τ is suppressed, the formulas of displacement satisfy the equations of equilibrium identically.

When the section is *variable*, the formal series in τ for the displacements become complicated; and as yet the author has not found opportunity to examine in any detail the convergence of these more involved developments. In the nature of the method, if we suppress the parameter τ in the series ordered according to degree of homogeneity in r and r_0 , these series are such that, when differentiated formally and substituted in the equations of equilibrium, terms of like degree annul each other; that is, the equations of equilibrium are satisfied in the usual formal sense.

Furthermore, there is a sense in which the formal series for the displacements yield approximate solutions. If we break off the series and discard terms of order higher than τ^{2k+1} , we obtain formulas of displacement which correspond to a total surplus body force per unit of volume of order τ^{2k} and a total surplus surface traction per unit of area of order τ^{2k+1} . Thus the truncated series correspond to a total surplus applied force of order τ^{2k} , whereas the total *given* applied force is at most of order τ^2 . If k is large and the rod is thin, it seems probable that when the surplus applied forces are removed, no sensible change occurs in the displacements, and that we have a physical solution of our problem.

† Cf. Paper I, § 16.

The present paper is concerned with the restricted theory * of strain, and the material is assumed homogeneous and isotropic.

Our assumption of developability in the "neighborhood of the axis" does not permit the satisfaction of the most general type of boundary condition at an end. If the traction is distributed in the manner we have specified, our solutions are *exact*; if it is distributed otherwise, without ceasing to be equivalent to a resultant of the type Z_h , the solution represents the state of the rod with sufficient approximation at all points which are not too close to the end.

The method of series affords a solution of a problem which has interested elasticians since Poisson and Cauchy. It is applicable to a variety of one- and two-dimensional problems—with thickness either uniform or variable—and its usefulness may extend to hydrodynamics, electricity, and electro-magnetism. For further indication of the applicability of the method,† the reader is referred to the paper on "Circular Plates of Constant of Variable Thickness."

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.,
March, 1923.

* Cf. Love, p. 57, on the general theory of strain.

† At the time of correcting proof the author is able to supply references to his recent notes in the *Comptes Rendus*, Vol. 178, Feb. 11, 1924, p. 619, and Vol. 179, Aug. 18, 1924, p. 381. These notes are concerned with the applications of the method of series to the thick rectangular plate and to the beam of rectangular section.

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